The Z-Relation in Theory and Practice

by

Jeremiah S. Goyette

Submitted in Partial Fulfillment

of the

Requirements for the Degree

Doctor of Philosophy

Supervised by

Professor David Headlam

Department of Music Theory

Eastman School of Music

University of Rochester

Rochester, New York

2012

Curriculum Vitae

Jeremiah Goyette was born in Lubbock, Texas on July 16, 1976. He graduated with a Bachelor of Music Theory, *magna cum laude*, in 2005 from Texas Tech University. While working on his Bachelor's degree, he studied Violin Performance for one year at the Conservatorio della Svizzera Italiana in Lugano, Switzerland, studying under the maestro Carlo Chiarappa. He began his graduate studies in 2005 at the State University of New York at Buffalo, where he studied late-nineteenth-century chromaticism with Professor Charles J. Smith. After two years at Buffalo, he transferred to the Eastman School of Music, University of Rochester, where he pursued research on pitch-class set theory and computation under the direction of Professor David Headlam.

Acknowledgements

The completion of this dissertation would not have been possible without the generous support of many people. First, I offer deep and sincere thanks to the members of my dissertation committee: Dave Headlam, Robert Hasegawa and Ian Quinn. They have each helped my writing and ideas take shape, and have been most encouraging. I would like to thank my advisor, Dave Headlam, for his constant and thoughtful assistance throughout the whole process. Not only did he draw my attention towards the pertinent work of others, but he also allowed me space to explore my own ideas while acting as a springboard for new ones. I also cannot give enough thanks to Robert Hasegawa, who not only helped with many of the finer details of the theory, but also gave invaluable suggestions on the prose and organization of the thesis.

I would also like to give my thanks to José Oliveira Martins and Douglas Ravenel for their guidance when I was just starting the project, and to Robert Morris, for his suggestions on several nuts-and-bolts aspects of the theory and his observations on the history of the z-relation. Additionally, I would like to thank my colleagues, Sebastiano Bisciglia and Justin Lundberg, and the members of Headlam's "Super Computer Club" for their critique of my ideas in Chapters 2 and 3 as they were in their early stages, and Professor Headlam for having introduced me to computer programming, without which this project would not have been possible.

I am grateful for all of the support that I received from the Eastman community, from both the theory faculty and the graduate students. There are too

many people to list here who have been influential to my overall philosophy of music, and who have encouraged me to press forward. However, in particular, I would like to thank Jonathan Dunsby, Steve Laitz and Marie Rolf for awarding me the opportunity to be a part of such a vibrant community.

Finally, I thank my entire family for their love and encouragement, and above all, my wife Kristi for her immeasurable support in this and in all my endeavors. I thank her from the bottom of my heart not only for her support, both mentally and emotionally, but also for keeping me motivated and moving on target. Without her, this dissertation may have never seen its completion.

Abstract

Though the z-relation—the relationship whereby two sets not related by transposition and/or inversion have the same interval-class content—has been one of the core concepts of pitch-class set theory since its inception, the principles underlying the relationship have to a large extent remained obscure. However, new information is emerging. First, recent work on the Fourier transform for pitch-class-set analysis (in particular, by Ian Quinn, which builds on work by David Lewin) provides new information concerning the subset structure of z-related sets. Second, as recognized by Clifton Callender and Rachel Hall, the z-relation is an instance of what crystallographers call *homometry*, which has been written about extensively. The aim of this study is to utilize these new means to present a comprehensive description of the z-relation that addresses the criteria upon which the z-relation relies as well as the ways in which z-related sets interrelate.

Building upon algebraic formulas suggested by crystallographers, I develop an approach that describes z-related pairs in terms of formulas that correspond to subset properties illustrated by the Fourier transform. In addition to devising my own formulas, I show that the formulas can be extended with a method called 'pumping' (O'Rourke, et alii), which adds pitch-classes to preformed formulas. Together, the formulas and pumping lead to a general theory of the z-relation that not only describes the relationships between the sets of a z-related pair, but also those between z-related pairs of different cardinalities.

Since it exposes transformational relationships, the algebraic approach lends itself well to the analysis of music that involves z-related sets, whether twelve-tone music based on rows with discrete hexachords that are z-related, or other non-serial music that prominently features z-related sets as motivic harmonies. In my own work I use the theory to analyze twentieth-century works by composers including Schoenberg, Berg, Carter and Berio. Overall, the theory exposes that transformational networks involving z-related sets are not only possible, but are altogether relevant to the analysis of music that involves such harmonies.

Table of Contents

List of Tables

List of Examples

Introduction

The z-relation is the relationship whereby two pitch-class sets have identical interval-class content but are not related by either transposition or inversion. In music theory, the phenomenon was first discovered by Howard Hanson and David Lewin contemporaneously in 1960, and came about during a period in which music theorists sought ways to categorize chordal equivalence. Generally speaking, there were two kinds of equivalence identified during this time: one by transposition and/or inversion, and one by interval-class content. Most adopted the former; however, Allen Forte (1964) defined the 'pitch-class set' (set class) by interval-class content, grouping together sets with identical 'interval vectors' even if multiple transposition/inversion classes fell under a single heading.[1] However, in response to John Clough's (1965) criticisms of his usage of interval-class content to define the set class, Forte reworked his theory of set-complexes in *The Structure of Atonal Music* (1973) and defined the set class instead by transposition and inversion, though keeping the notion of interval-content equivalence as a central component of his theory. Since then the z-relation has become a standard, if not completely understood, concept in pitch-class set theory.[2]

---

[1] Throughout both his 1964 and 1973 works, Forte often used the term 'set' loosely to indicate both a collection and a class; however, he also used 'set class' as early as 1966. To avoid confusion, I use 'set class' to indicate an equivalence class, and 'set' to indicate a collection.
[2] See Chapter 1 for a history of the origin of the z-relation, and about Clough's criticisms of Forte's original use of the interval vector as set-class determinant.

Indeed, though the z-relation is well-known, there remain many unresolved questions about it, including its mathematical associations and origins and its significance as a musical relationship. One particular problem stems from the definition of the z-relation itself. Being defined by the equivalence relation based on shared interval-class content, the z-relation is type of equivalence: for any pairs of pitch-class sets, if the defined criteria are met (same interval vector but not related by T or I) then the z-relation is present. However, in contrast to the other usual equivalences that are based on the canonical twelve-tone operators—transposition, inversion and multiplication[3]—the z-relation is not defined by some action (an operator); rather, it only reflects a shared quality or state of being (interval content) between two sets.[4] Thus, while it indicates an abstract set relationship, the z-relation (as it is typically understood) says nothing about how one z-related set can be transformed to another.

In the long run, the fact that the z-relation is defined solely as the result of an equivalence relation (as it is in Forte 1964 and 1973) has been problematic for the application of the z-relation in music analysis: not only does it describe a relationship not based on any common set operators, but the relationship that it does describe

[3] O'Connell 1962 was perhaps the first music theory work to recognize M (multiplication by 5), though it was Howe 1965 that later defined it as one of the three twelve-tone operations, T, I and M. Morris (1982 and 1987) showed that these three operators form a group, and are thus 'canonical'. Lewin 1987 also used the term 'canonical,' though he used it more loosely to indicate the existing operations within a given space, whatever they may be.

[4] This distinction between *being* and *action* corresponds to what Lewin described as "Cartesian thinking" and a "transformational attitude"—that is, a perspective that focuses on the objects, versus one that focuses on motions between objects. (Lewin 1987, 158-159, cited in Hook 2007, 172).

seems to say little about our musical experience. Critics of the z-relation, such as John Clough and George Perle, have maintained that the z-relation is musically unintuitive since it cannot be readily identified by ear (Clough 1965 and Perle 1982 and 1990). Since the z-relation is (as they see it) an abstract relation that is arguably difficult to hear, they suggest that there is little to be gained by observing that two pitch-class collections are z-related.

Lurking behind much of the discourse on the z-relation by both proponents and critics, however, is a tacit presupposition that there is no more to the z-relation than that by which it is defined—that is, that the z-relation merely indicates the condition of shared interval-class content. Only a handful of studies (including in particular, Morris 1987, 1990 and 2001, and Soderberg 1995) have investigated whether or not the z-relation rests on any underlying phenomena, whether the z-relation is indicative of other common aspects shared by sets, or how the z-related sets can be enumerated. Rather, since mainstream theory has generally been concerned more about whether identical interval-class content is actually audible, many have tended to accept the z-relation at face value, treating the definition as both the cause and the effect. Since the z-relation was associated with the notion of 'set similarity' already in Forte's early work—with the idea that the z-relation indicates a high degree of similarity between two sets of different set class—many have simply accepted its status as a type of equivalence without needing to inquire about its underlying mechanics.

Without some understanding of what conditions make the z-relation possible, little can be said about two z-related sets other than that they share the same interval content. However, there is a good deal of music that has fundamental motivic harmonies that are z-related sets, and that exploit particular features of the z-related sets to create notable harmonic structures. For such music, simply saying that two sets are z-related is not enough. Instead, what is needed is a mechanism for showing the ways in which z-related sets relate. Certain patterns seem to unfold with z-related sets based on identical subsets; however, without a system for describing the relationships that stand between a pair of z-related sets, it is difficult to make absolute statements about these patterns.

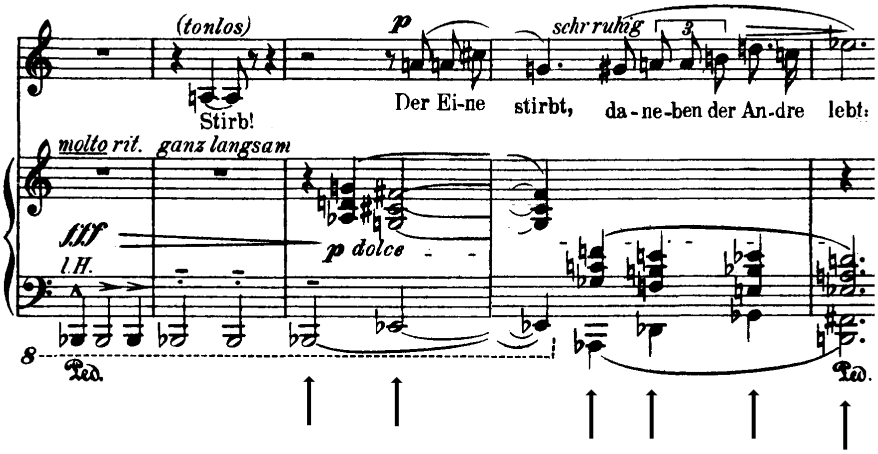
As an example of a notable usage of z-related sets, let us consider a passage from Berg op. 2, no. 4, which has alternating z-related sets in the piano part (see Example 1).[5] In the passage, the six chords in the piano part marked by arrows (Ex. 1a) form a sequence of alternating z-related sets formed by two convergent patterns: a chromatically descending [016] trichord in the right hand, and ascending interval 5s in the left hand. Except for the last chord, which has an added F♯, every other chord is one of the two all-interval tetrachords (4-z15 and 4-z29), [0146] and [0136], which are voiced as dominant-♯9 and -13 chords, respectively.

---

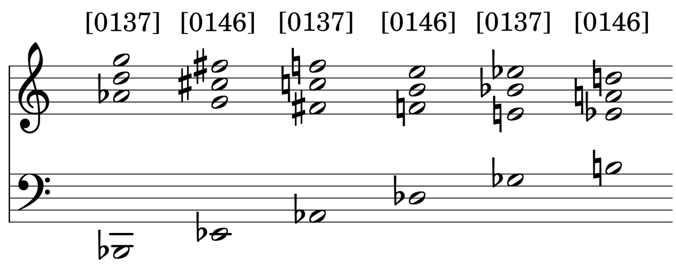
[5] This passage actually appears in an earlier piece, *Le Gibet*, the second movement of Ravel's *Gaspard de la Nuit*. Schoenberg cites the chord in his *Harmonielehre* (1978, p. 420). The Berg passage is described in Headlam 1996, p. 419, n. 102, where he observes that the all-interval tetrachords may be related by a transposing a single pitch by T_6, holding either a [016] or [026] subset in common, or, as in the setting, when the total displacement (+5, -1) is by 6.

Example 1: Sequence of all-interval tetrachords in Berg, op. 2, no. 4, mm. 18-22.

a)



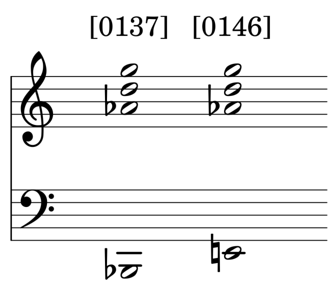
b)



Because of the strictness of the pattern, the example invites a transformation-based explanation on how the two z-related set classes interrelate. What concerns us here is not that the two set classes oscillate sequentially, which is simply a result of the transposition levels (T_{-1} and T_5) of the upper and lower voices; rather, the point of interest is the motion between the two z-related set classes. The passage shows that an instance of either all-interval tetrachord transforms onto an instance of the other all-interval tetrachord simply by a transposition of the [016] subset down a step and the other pitch up interval 5. If we transpose the second chord up a half step, then the

motion from the first chord to the second becomes even simpler (see Example 2): the two all-interval tetrachords both share the [016] subset while the other pitch is transposed by T_6. The same could be done to other subset; that is, the same T_6 motion can be applied to either the singleton or the [016] subset, and it always transforms the z-related set to an instance of the other z-related set class.

Example 2: The first two chords of the passage, with the second chord transposed up a semitone to reveal a single tritone motion.



There have been a few recent authors who have observed this particular relationship (among other relationships) between the all-interval tetrachords. Writing in reference to the music Elliott Carter, Guy Capuzzo (1999 and 2004) and Adrian Childs (2005) have made many observations about the relationships between all-interval tetrachords, and have shown that there are several contextual transformations that consistently change one all-interval tetrachord to another, including ones like in Example 2. Nevertheless, despite their work, there are a number of questions that remain. For instance, do such transformations exist for all z-related sets? If so, can we generalize these transformations for all of the z-related sets, or does each z-related pair have its own unique transformations?

There are also some other broader questions about the z-relation that we must grapple with if we are to gain an understanding of the principles upon which the z-relation relies.

1. What criteria must be met in order so that two sets of different set classes share the same interval-class content?
2. Is it possible to enumerate the z-related sets; in other words, is it possible to calculate the number of z-related sets without having to count them either by hand or with a computer?
3. Do all groups of z-related sets satisfy the same criteria, or are there different types of z-relation?
4. Given a z-related pitch-class set, how can one derive the z-related partner?
5. How can one tell if a random pitch-class set is potentially z-related set?

Up until recently, the best answers to these questions have come from Robert Morris and Stephen Soderberg. Morris (1982, 1987, 1990 and 2001) has uncovered many interesting aspects about z-related sets. For instance, he has shown that set unions of particular subsets can generate each of the 15 pairs of z-related hexachords, as well as the all-interval tetrachords, thus providing at least partial answers to questions 1, 3, 4 and 5 above. While Morris has developed both descriptive and generative theories about the z-relation, Soderberg's contribution, the Q-grid, is generative by design, in that it provides a means for generating z-related sets from scratch. Furthermore, Soderberg has shown with the Q-grid that some z-related sets are related by a dual inversion, exposing for the first time in music theory a distinct transformational

relationship between z-related sets. There are many fascinating aspects about the Q-grid, but overall it only provides tentative answers to questions 1, 2 and 4 above.

The theory of the z-relation has recently taken a turn, however, driven by an influx of new information beyond that provided by Morris and Soderberg. First, recent work on the Fourier transform (in particular Quinn 2007), has illustrated that sets that are z-related have identical outputs (Fourier coefficients). One consequence of this observation, as I show, is that z-related sets share cyclic-collection subsets.

In addition to the Fourier transform, there has been a recent recognition (first by Clifton Callender and Rachel Hall in 2008) that there are analogous studies on the z-relation in other fields—in particular, crystallography (the study of the atomic structures in minerals and other substances), where they are called *homometric* sets. In crystallography, homometricity is a relatively central issue, and thus there have been numerous studies on the topic.

My work begins with this recent information concerning the z-relation. From crystallography, I adopt certain generative models and formulas (based on the underlying conditions of homometricity) that generate homometric sets, or that determine if a set is potentially homometric. However, since much of the work in crystallography involves sets in 3 dimensions (atomic structures are 3-dimensional), not all of the theory on homometric sets can be imported into music theory (where sets are regarded in a periodic 1-dimensional space). Nevertheless, there are many parts of the theory on homometric sets that are generalized so that they apply equally in a variety of spaces, some of which can be absorbed into music theory without too

much trouble. One of the goals of this project is to translate some of the observations made by crystallographers to a musical context. Though crystallographers have made some observations that require little translation, such as the formula for the all-interval tetrachords, there is overall a vast difference in attitude between the theory on homometric sets and that of the z-relation in music theory—a direct result of the difference in aim between the two disciplines. While crystallographers are concerned with successfully translating intervals (interatomic distances) into sets (atomic structures, see Chapter 2), music theorists are concerned with what the z-relation indicates about already-known pitch-class collections. Accordingly, my goal is to restate these observations from crystallography in terms that are compatible with current music theory.

I also extend some of the ideas that are presented in the crystallographic literature. Though while crystallographers have provided some formulas that correspond directly to z-related sets in mod12, their work covers a broad spectrum of spaces, and consequently does not cover all of the z-related sets in mod12. Such a limitation is expected: crystallographers are not necessarily concerned about mod12 space the same way as music theorists are. My work thus extends some of the formulas from crystallography to be able to account for all of the z-related sets in mod12, and to create transformations that apply to all of the common z-related sets.

The formulas that I adopt show that z-related sets are formed from unions of particular subsets. Like the work of Morris, these formulas derive the z-related sets from subset unions involving cyclic-collections. However, the formulas presented

here extend Morris's work, in that they show more precisely how the subsets are situated in relation to one another, and they provide an exhaustive means for outlining the relationships between all of the various instances of a pair of z-related set classes. In addition, my formulas are based on criteria that are generalized among the various modular spaces, as well as even continuous modular spaces (as described by Callender 2007). Thus, like Soderberg's Q-grid, the present theory is able to describe the relationship between z-related sets in wide array of spaces besides mod12 pitch-class space.

From the formulas I develop a number of transformations that change a z-related set to an instance of the z-related-partner set class—what I call *z-transformations*. The z-transformations, which transpose or invert certain subsets while holding others in common, provide a means for describing concrete relationships between two sets that are z-related. Though the z-transformations are contextual—in that they affect certain pitch classes depending to structure of the set—they, like the canonical TTOs, form a group (in the mathematical sense), and consequently can be used in combination with the usual operators, including multiplication. In my own analyses, I show that combining the z-transformations with T_n extends the domain of the z-transformations to all of the instances of the two set classes of a z-related pair—a feature that is necessary for music analysis. Overall, the z-transformations provide a strong framework for investigations into transformational relationships between z-related sets.

The four chapters that follow align into three broader headings: Chapters 1-2 cover the history of the z-relation, as well as other current research; the second half of Chapter 2 and Chapter 3 develops my ideas, which are inspired by recent work presented in Chapter 2; and Chapter 4 covers some applications of my theory for music analysis.

Chapter 1 outlines the origins of the z-relation, and other subsequent research up through Soderberg 1995. It begins with a discussion on Lewin's discovery of the z-relation as a byproduct of his interval function, on Forte's adoption of the z-relation in his first work on set theory, and on the resulting critiques that followed. I then turn to look at other observations made by music theorists—in particular, Morris and Soderberg. As I show, though both Morris and Soderberg made a number of key observations concerning the z-relation, their work provides only partial explanations of the mechanics of the z-relation.

In Chapter 2, I introduce new research on the z-relation. After explaining how homometric sets are relevant to crystallography, I review some of the key observations in crystallography on homometric sets (which are actually old observations, but new to music theory), and some of their formulas for homometric sets. The formulas presented here are the basic building blocks for what is developed later in Chapter 3. Then, I review the Fourier transform as a means for pitch-class-set analysis, focusing on the theories proposed by Lewin and Quinn on the Fourier transform. Finally, I extend their work to show some additional features of z-related sets.

The entirety of Chapter 3 is original work, or at least an original extension of other ideas. In this chapter, I lay out the bulk of my theory: I propose a method for building z-relation formulas from scratch; I discuss the z-transformations and their group structure; and, I explore pumping (a method for generating z-related sets, introduced by O'Rourke, et al., 2008) and propose some original pumping algorithms.

Chapter 4 is devoted to analytical applications of the theory proposed in Chapter 3. Included are several examples from the music of Elliott Carter, an example from Berio's *Sequenza IXb*, and several examples from Schoenberg's 3rd String Quartet, opus 37—which uses z-related hexachords. Many of the examples focus on the all-interval tetrachords. With one example from a work by Carter, I make a comparison of my work with the work of Capuzzo. In regards to Schoenberg's quartet, one of the few twelve-tone works by Schoenberg that do not use combinatoriality in the usual ways, I show that the order of the row aligns with the structural properties of the two z-related hexachords in the row, which leads to possible invariances between row forms. As I demonstrate, these invariances can be traced using z-transformations. Finally, Chapter 4 closes with a discussion on using the Fourier partition as a heuristic for identifying z-related sets.

1. History of the Z-Relation in Music Theory

1.1 Origins of the z-relation

Though the z-relation is most commonly associated with Allen Forte, as he gave it its name, he was not the first music theorist to discover it. There were already written accounts of the z-relation in music theory in two independently developed works published the same year: Hanson 1960 and Lewin 1960. Howard Hanson was perhaps the first to acknowledge the z-relation, but his book was not as well circulated in music theory circles.[1] The line, rather, seems to begin with David Lewin, and continue through Donald Martino to Allen Forte. As modern pitch-class set theory develops primarily from the Babbitt tradition of twelve-tone theory (from Babbitt to Lewin and Martino, then to Forte), so too does the z-relation. Accordingly, the following historical overview traces the origins of the z-relation starting with Lewin and continuing to Forte, with early criticisms from John Clough and George Perle.

1.1.1 Lewin and the interval function

The first two articles of Lewin's career (Lewin 1959 and 1960) are an extension of the twelve-tone theory developed by Milton Babbitt, with whom Lewin studied at Princeton. In these articles, Lewin adopted many of the Babbitt's concepts, such as the notion of octave-equivalent pitch classes, and that the primary

---

[1] Cohen suggests that credit should be given to Hanson since his book probably took more time to write than Lewin's article (Cohen 2004, 75).

equivalence classes are the canonical twelve-tone operators, transposition and inversion; however, instead of focusing on rows, he focused specifically on "collections of notes", or unordered sets of pitch classes, an idea that derives from Babbitt's "source set." Babbitt (1955), as well as Martino (1961), used the term "source set" to describe unordered row segments, from which certain qualities of rows, such as combinatoriality, could be determined.[2] Lewin's collections, however, are distinct from Babbitt's source sets, in that for Lewin a collection could exist outside of the context of a twelve-tone row. In this way, Lewin's first two early works act as a bridge between twelve-tone theory and pitch-class set theory: on the one hand, these two articles derive some of the basics of pitch-class set theory, such as defining a set as an unordered collection, and generating classes of pitch-class sets through equivalence by 'transposition or inversion followed by transposition'; on the other, these articles forecast some later developments, such as the Fourier transform, which since its reappearance, first in Lewin 2001 and then in Ian Quinn's dissertation (2004), has come to significantly alter the debate for many areas in pitch-class set theory.

In these two early articles, Lewin presented for the first time and elaborated upon his 'interval function,' one of the principal musical concepts developed throughout his career. The interval function foreshadows Forte's interval vector, and

[2] Schuijer (2008, 96) points out that Babbitt's use of the term "source set" dates back to his 1946 doctoral thesis (Babbitt 1992, 104-116). The idea of focusing on small intervallic groups comes from Schoenberg, especially his analysis of his Opus 22 songs (see Schoenberg and Spies 1965), which was later formulated by Perle as the "basic cell" (Perle 1962). The earliest formulation of the pitch-class sets is by Berg's pupil, Fritz Heinrich Klein (Klein 1925, translated in Headlam 1992).

is the backbone of Lewin's later transformation theory. In Lewin 1959, the interval function is defined as the following:

> For every integer i between 0 and 11 inclusive, let m(i) (read "m of i") be the number of pairs of notes (x,y) such that x is a member of the collection P, y is a member of Q, and the interval- between x and y is i. The function m will be called the interval function between P and Q.

Essentially, the function m(i) tallies the intervals found between two collections of elements. Though by 'interval' Lewin is referring here to the 12 directed pitch-class intervals, the function is generalized so that it is exists in multiple spaces.

Through the interval function, Lewin identified in his 1960 article what is now called the z-relation. He observed that certain pairs of sets not related by transposition or inversion have the same intervallic content—defined as the interval function between a set and itself. For a given pair of sets P and Q, he lists five cases in which the two sets have the same intervallic content (Lewin 1960, 99-100):

1. Q is a transposition or inversion of P.
2. P and Q are six-note collections and Q is a (transposed or inverted) form or the complement of P.
3. P (or Q) is a (transposed or inverted) form of the tetrachord C, D, F, F♯, and Q (or P) is a form of the tetrachord B, C, D, F♯.
4. P and Q are forms of the pentachords B, D, F, F♯, G, and F, G, A♭, B, C.
5. P and Q are forms of the pentachords B, C, D, E, F, and C, C♯, D, E, G and these are the only possibilities.

The first case is what can be called the 'set class' case, while the other four cases all deal with instances of the z-relation. Cases 3-5 identify specific pairs of z-related sets.

There is partial redundancy between cases 1 and 2, since the 'Q as transposed or inverted form of P' in case 2 is already covered by case 1; but the redundancy in case 2 helps to outline what is known today as the generalized hexachord theorem—the theorem that states that complementary hexachords are either of the same set class or z-related. Lewin's case 2, in fact, is regarded to be the first iteration of the generalized hexachord theorem in music theory.[3]

The seven- and eight-note z-related pairs are not mentioned in any of the five cases; however, Lewin does account for them later in the discussion, when he invokes his Formulas I and II, which show the relationship of the intervallic content between a set and its complement. As his Formula II indicates, if P is a set of cardinality x, then the intervallic content of the complement P' is:

$$p'(i) = 12 - 2x + p(i) \qquad \text{for each interval } i$$

which can also be written as

$$p'(i) - p(i) = (12 - x) - x \qquad \text{for each interval } i$$

In other words, the difference between the interval function of a set to itself, and that of its complement, is the difference between the cardinalities of the two sets. Because of this formula, it clearly follows that if a pair of sets falls under cases 3-5, then the complementary pair will also have the same intervallic content.

[3] Ballinger, et al. 2009 gives a brief history of the hexachord theorem and its various proofs in both music theory and in crystallography. Though Lewin is the first music theorist to iterate the hexachord theorem, crystallographers observed the theorem several years prior to Lewin (Patterson 1944)—observations of which I do not think Lewin was aware. In music theory, several theorists after Lewin provided mathematical proofs of the theorem, such as Kassler 1961, Regener 1974 and Wilcox 1983.

Despite his claim in case 5 that these five cases are the only possibilities, Lewin's list is actually incomplete: it lacks one pentachordal pair ([01248] and [03458]). There is no mention of the pair elsewhere, and is presumably an oversight. Curiously, this pair and its complemental pair are the only two pairs in mod12 that are based on the cyclic collection {0, 4, 8}, as opposed to the cyclic collection {0, 6} (see Chapter 2).

The hexachord theorem, as stated in case 2, had a significant impact on future conceptions of twelve-tone harmony. As Schuijer (2008) observes, Lewin (following Babbitt) came to believe that the hexachord played a central role in defining harmony in Schoenberg's twelve-tone music. In reference to Schoenberg and the hexachord theorem, Lewin writes (Lewin 1960, 100, cited in Schuijer 2008, 97):

> An interesting application may be made here to the hexachordal pieces of Schoenberg, and to similarly composed music. If a piece is based on a hexachord P and is so constructed that, in general, some form of P is always being played against the complementary form—P being distinguished from its complement at any given point, e. g. by instrumentation, melody-accompaniment, etc.—and if the order of notes within the hexachords is not too strictly constrained, then the interval function m(i) between P and P′ will probably exert an aural effect in the long run. Since m(i) represents the "total 'contrapuntal' relation" between P and P′, of course it also represents the "total 'contrapuntal' relation" between any transposed or inverted form of P or P′ and the corresponding form of P or P′. In this sense, the "total potential counterpoint" in any section of the piece is always the same. And, in the sense of our previous discussion, this "total potential counterpoint" uniquely characterizes P, P′, and their transposed and inverted forms. This may explain in what sense a hexachord has an aural "identity" in such a piece.

Since the intervallic content of two complementary hexachords is always the same, each twelve-tone work carries a sort of signature harmony, in that the hexachords "place certain constraints on the harmonic idiom of the work" (Schuijer 2008, 97). Naturally, not all twelve-tone pieces exhibit clear hexachordal partitionings; but because Schoenberg generally treats the hexachord as a compositional unit (particularly since *Suite*, op. 25, and onwards), such references to a general harmony seem appropriate for his music.[4] Lewin's claim may be over-reaching, however, since Schoenberg never discussed such ideas. Clearly, though, as shown most thoroughly by Mead 1985, the harmonic identity of the row, in whole and in part, was paramount. Ultimately, the idea that intervallic content is a coherence-providing ingredient in twelve-tone music helped to give rise to the notion that intervallic content could be thought of as a measurement, or like Forte, as an equivalence class.[5]

1.1.2 Forte, the interval vector, 'Z'-forms and Clough's criticism

Forte's set theory, appearing first in his 1964 "A Theory of Set-Complexes for Music," drew largely on the work of David Lewin and Donald Martino, both of whom studied under Babbitt at Princeton. While Forte never studied directly under Babbitt, his work can nevertheless be considered part of this line of Princeton twelve-tone theory, since it adopted many of its core principles. The article was the first iteration of a theory that remained fairly consistent throughout Forte's career,

---

[4] Using Schoenberg's sketches, Hyde 1980 shows some ways in which Schoenberg used hexachords (the "basic set") to create rows in op. 25.
[5] Forte 1964, Teitelbaum 1965 and Regener 1974 (among others) used the interval vector to determine similarity between pitch-class collections.

containing concepts such as superset/subset relations, as well as similarity relations based on the interval vector. However, though this article was original in many respects, a good portion of the technical apparatus was imported from Martino. For example, Forte's list of set classes had already appeared in Martino 1961, which, using Babbitt's terminology, Martino called "source sets." Also, Martino's list of set classes already included (among other information) 6-digit interval vectors and asterisks indicating the z-relation.[6]

One major difference between Forte's early theory of set classes and that of Lewin and Martino is that Forte did not define the set class by transposition and inversion, but instead by intervallic content. This reformulation of the definition of the set class can seen as a bold move by Forte, since it countered what were by then mainstream definitions developed in twelve-tone theory. Indeed, Forte later modified his original version after incisive criticisms in a review by John Clough (1965), which urged for a return to a definition of set class based on transposition and inversion. Nevertheless, after changing back in his later works to transposition and inversion, Forte continued to keep 'intervallic content' as a fundamental component of his theory.

After an initial discussion detailing the basics of what defines a pitch set and an interval class, Forte's 1964 article introduced the 'interval vector'. Forte's interval vector was not an original creation; rather, it was identical to Martino's interval

---

[6] Martino only adds asterisks to the pair [0146] and [0137] (Table 2, p. 237). The list of hexachords also indicates the z-relation, though implicitly, in that it is ordered by interval vector (Table 1, p. 229).

content, differing in name only.[7] Martino's interval vector had one particular advantage over Lewin's interval function: it is shorter and more concise. Though Lewin never wrote out interval vectors in the 1960 article, we can assume, judging from the nature of Lewin's interval function, that a vector based on the interval function would be twelve digits long. Martino's vector, on the other hand, was only six digits long. It is, of course, this shorter form, which Forte designated as the 'interval vector' (though actually an "interval-class vector"), that eventually became the industry standard.

After covering the interval vector, the 1964 article proceeds to a discussion on the set class, which includes Forte's first list of set classes. The 1964 list of set classes shares many of the features of his later, more accepted list of set classes (Forte 1973, e.g.), such as the overall organization and ordering of the set classes (by interval vector), and the use of the abbreviated Forte labels (3-1, 7-35, etc.). However, in contrast to his later writings, the 1964 version defined the set class by interval vector, and thus only counted a total of 201 set classes, instead of 224.[8] Each pair of z-related sets (using his later terminology) was counted as only one set class: one of the sets of each pair of z-related sets was considered as the representative set and included in the list, while the other was left out of the list entirely (even for the hexachords). In the

---

[7] Though Forte's use of the term 'vector' conflicts with the standard definition of the term in mathematics (where it is a line with direction and length), it agrees with the standard usage in computer science, where a vector is a list of objects (numbers, characters, etc.). That is, the interval vector is a list of integers that represent the number of instances of a particular interval class within a set.

[8] Concerning Forte's use of the interval vector as set class determinant in the 1964 article, see also Schuijer 2008 (pp. 89-93).

list, he noted such cases with a ‘Z’ preceding the set class number (such as, the hexachord ‘6-Z28’). In this early list, a set class number with a ‘Z’ thus indicated one interval-vector class with two members.

Despite that Forte was aware that there were these pairs of sets with the same interval vectors, having even cited Lewin 1960 as the source—in fact, it is when citing Lewin in footnote 10 where Forte first used the ‘Z’ indication outside of the lists, calling such sets ‘Z-forms’—Forte gave no explanation to his methodology, or to why one z-related was included in the list and not the other.[9] It appears, however, that he chose the set most compressed to the left, counting from the left. Further complicating matters, there was an unfortunate typo in the list of tetrachords: the set class 4-14, which should have been [0237], was instead written as [0146].[10]

In the 1964 article, not only were the set classes determined by interval vector, but Forte also ordered the list of set classes by the interval vectors. Martino also ordered the hexachords (though not the other cardinalities) by interval vector, but only after dividing the hexachords into four groups, depending on the number of ic6’s in the set.[11] Forte did no such filtering, but instead only ordered each cardinality of set classes by interval vector. For each cardinality, the list begins with the sets with the

---

[9] For many years, the meaning of ‘Z’ was not publicly known. On November 17, 2004, however, Forte revealed the name at a conference at the Orpheus Institute in Ghent, Belgium, saying that the Z indicated “zygotic”, which in biology refers to a fusion of two reproductive cells (Schuijer 2008, 98).

[10] Forte 1964, 146. He does give the correct interval vector—that is, the one for the set class [0237].

[11] Though Martino does not point out the connection, his method of filtering the hexachords by the number of tritones is strongly reminiscent of Hindemith’s chord classification in *The Craft of Musical Composition* (Hindemith 1942).

highest number of intervals towards the left of the vector (with the most ic1s). To put it most simply, the ordering is a result of reading the interval vectors as 6-digits numbers, and arranging them in descending order. For example, the set [0237] (with the interval vector <111120>) comes before [0137] (<111111>) since 111120 is a greater number than 111111.

Basing the set classes on the interval vector creates some natural problems. First, in contrast to interval vector equivalence, it is generally argued that sets such as {0,1,4,6} and {0,1,3,7} are more distantly related than say {0,1,4,6} and {0,2,5,6}. Both pairs share the same interval vector, but the latter pair is related by inversion, whereas the former is not related by transposition or inversion. In this vein, one of the main criticisms that John Clough (1965) made in his review of Forte's article was that the interval vector is musically unintuitive. Clough argued that any notions of 'pitch-set' (set class) should be natural for a musician, and not overcomplicated by theoretical constructs. He asserted that for any theoretical concept we should "reserve the right to evaluate a proposed relation at least partly, and certainly initially, in terms of our intuitive ideas of musical similitude" (1965, 164). That is, a useful and potentially successful definition of set class should be able to integrate with the already present theoretical knowledge and skills held by most musicians. As Clough argued, musicians are not necessarily able to recognize the interval-class content of chords, but they are very familiar with the notions of transposition and inversion, judging by the long history of these musical concepts. Moreover, to define set class

by interval vector leads to a more abstract notion of set class that is difficult to identify both visually and aurally.

Clough also claimed that the interval vector is inherently limited. As he pointed out, the interval vector is essentially a tally of the dyadic subsets in a set. Since it does not consider other subsets (trichords, etc.), the interval vector only gives an incomplete description of a set. Though Clough did not develop further the idea of trichord vectors and the like, it is clear that he was saying this: two sets are of the same set class only if *all* of the subsets are the same—a statement that, in the end, corresponds exactly to the transposition/inversion definition of the set class. Though it may counter the definition of the z-relation, Clough's idea of trichord vectors (and tetrachord, etc.) is an intriguing idea, and was discussed later by Lewin.[12] Nevertheless, Clough's critique assumes that since the interval vector is incomplete, the z-relation must be insignificant—an assumption that I argue to be incorrect.

In a response to Clough's review (Forte 1965), Forte launched a defense based on the idea that Clough had confused 'equivalence' with 'similarity' (174). Forte argued that Clough's appeal to the musically intuitive was dubious and arbitrary, and that even inversion is "a mathematical abstraction, one which may be relevant to a particular musical situation, or again may not be of interest" (175). As Forte argued, the relationship between two sets that share the same interval vector is not necessarily any more abstract than that between two inversionally-related sets, such as the major and minor triads. Forte also added that as an equivalence relation, the interval vector

[12] Lewin (1987, 106-108) mentioned the possibility of "M-class vectors," which derive from his EMB function.

is just as mathematically sound as inversion, even though they produce two different set universes. In his initial defense Forte budged little, claiming that Clough was unjustly attached to inversion; nevertheless, in his later 1973 work, Forte ultimately conceded and took Clough's recommendation to define the set class by transposition and inversion.

In later incarnations of Forte's set-class list, such as in Forte 1973, the list included the 23 set classes that were originally omitted. However, in his later list he altered the logic of the ordering. Instead of inserting the new set classes into the list, placing them next to the sets to which they are z-related, he added the new sets to the end of the list. For example, set class [0146] was not inserted after [0137] (4-Z15), which would have made it 4-Z16; nor did he choose to define two forms for 4-Z15, such as 4-15a and 4-15b. Instead, the set class was added to the end of the list of tetrachords, and thus it took on the label 4-Z29. In addition, in the 1973 list, Forte switched the label numbers for [0137] and [0146], making [0146] the set 4-Z15 and [0137] the set 4-Z29. For cardinalities 5-7, which have multiple z-related sets, Forte ordered the new sets by interval vector, and then added that ordered group to the end of the list. In effect, by not inserting the new sets into the positions that correspond to the correct ordering by interval vector, the overall logic of ordering by interval vector was disrupted. The reasons for this ordering are not known, but it is presumable that he did not want to have to change the set class labels that he had already designated; otherwise, there may have been confusion as to which version of the list a label was referring.

Though the Forte labels became common parlance for many theorists and were thoroughly memorized by many of Forte's original students, there have been several published set-class lists with different orderings.[13] Morris's (1987) list replicates Forte's 1973 ordering, with set classes [0146] and [0137] (4-Z15 and 4-Z29) switched. John Rahn's (1980) algorithm for ordering the set classes, which is probably the most common today after being adopted by Straus (2000), did not consider the interval vector. Instead, it ordered the set classes so that the list begins with the set class most compressed to the left (in normal order and prime form), counting from the left. Rahn's algorithm is basically the same as Forte's, except that it is applied not to the interval vector, but to the set itself—that is, his list treats the set classes as if they were numbers ([0123] would be, for example, 123), and orders them in increasing order.[14] Another popular ordering of the set classes comes from computer research in music theory: though he never published a list of the set classes, Alexander Brinkman (1986) added another way to order sets with his method of representing pitch-class sets as binary numbers.[15] After turning the set classes into binary numbers—where, for example, the set {0,1,4,6} becomes the binary number 000001010011—set classes can be ordered by merely arranging the binary numbers

[13] Martha Hyde, one of Forte's former students at Yale, often recounted stories in her classes at SUNY Buffalo about students who, when taking post-tonal courses with Forte, would commit to memory the set classes and their corresponding Forte labels.
[14] As noted by David Smyth (1983-4, 550-552), Rahn also uses a different algorithm for finding the 'normal form', which leads to some prime forms that differ from those in Forte's list. These are 5-20, 6-Z29, 6-31, 7-Z18, 7-22 and 8-26.
[15] Brinkman 1990 (p. 628, n. 5) mentions Daniel Starr (1978) and Bo Alphonse in connection with the binary model, and notes that he learned it from Leland Smith in 1974.

from lowest to highest. His method benefits from being simple and easily adaptable to computing. Nevertheless, despite the many alternate algorithms for ordering the set classes, Forte's labels still have currency in the field of pitch-class set theory.

1.1.3 Perle's criticism of Forte's pitch-class theory and the z-relation

In a 1982 letter, and in a 1990 review of Forte's pitch-class set theory (in particular, that of Forte 1973), George Perle gave a stinging criticism of pitch-class set theory as a whole, which included several critical points about the interval vector and the z-relation. Perle raised a number of points about pitch-class set theory. For instance, Perle argued that Forte's theory has no connection to the practices of composers that the theory is supposed to address, and suggests that Forte's theory, like Schillinger's theory (Schillinger 1941), is 'Martian' musicology (Perle 1990, 159-160). As Perle argued, there is nothing in the writings of Schoenberg that suggests that compositions from the so-called atonal period were conceived under the terms outlined by Forte. The theory of set classes developed from twelve-tone theory, and the early works of the second Viennese School were written before such theory had arisen. Perle asks, "[w]hy should these composers have made it so amply clear that they were aware of the structure and functions of the 12-tone row, but never have dropped even the vaguest hint that they were aware of any aspect of the 'general theoretical framework' that Allen Forte has discovered to be the basis for their atonal music?" (1990, 152).

Another point that Perle raised, which also still resonates today, is that underlying Forte's theory is the assumption "that the concept of the pitch-class set has

the same universality for all of non-12-tone atonal music as the concept of the 12-tone set has for music in the 12-tone system" (1990, 154). Even though there is little evidence that suggests that composers of early post-tonal music were thinking along the lines of Forte's theory, Forte nevertheless endows the set class with the same reverent status as that of the twelve-tone row. This is Forte's signature ideology, Perle argued, as "no one before Forte had ever suggested that everything that happens in an atonal piece must unfold a pitch-class set statement, just as everything that happens in a 12-tone piece unfolds a 12-tone set statement" (1990, 155). Though I will not address them here, such questions about pitch-class set theory have yet to be settled, and there is still an ongoing debate over the legitimacy of pitch-class set theory for music analysis.[16]

Though Perle's criticism of Forte's pitch-class set theory may not be immediately connected to the issue of the z-relation, the points he makes are nevertheless relevant to how the z-relation might be used in analysis. Since the z-relation requires a theory of pitch-class sets, it is necessary to be transparent about how pitch-class sets would be used in analysis. In the analyses in Chapter 4, I have tried to be clear about questions of segmentation, structural significance, and salience in the choice of z-related sets used.

In addition to his criticisms of Forte's pitch-class set theory, Perle also makes some critical remarks regarding the z-relation. First, if all of the 35 complementary

---

[16] Since Perle, the strongest attacks on pitch-class set theory have come from, in particular, Richard Taruskin—see, for instance, Taruskin 1980, which led to subsequent exchanges between Taruskin and Forte.

hexachordal pairs (each of them necessarily forming aggregates) have the same interval vector, "[w]hy should a special value be placed on the…fifteen whose two hexachords are neither inversionally nor transpositionally equivalent?" (1990, 166). In twelve-tone music, Perle argued, there is little to suggest that z-related hexachords were much considered by composers, especially since so much attention had be given by Schoenberg and Babbitt to the 20 non-z-related hexachords, and particularly the six all-combinatorial hexachords. Though it is true that little has been written about z-related hexachords, Perle is arguably over-reaching with this claim. Much of twelve-tone theory is indeed overshadowed by the notion of combinatoriality; however, it is incorrect to say that z-related hexachords were never a concern for twelve-tone composers. Schoenberg's 3rd String Quartet is a case in point (see Chapter 4), as is Berg's Violin Concerto and several other works.

Perle also argues that if the majority of z-related sets are hexachords (15 of 23 pairs), then would it not be more interesting to talk about the tetrachords or pentachords, since, at least theoretically, they should appear less often in music? That z-related hexachords would appear more often than the other z-related sets, however, is not necessarily a true statement, and is difficult to accurately assert; but it would be nevertheless a natural conclusion judging by Forte's own use of z-related sets in analysis. Because of some other theoretical concerns, such as his R-relations and K- and K_h-complexes, Forte tended to prefer hexachords in analysis. Although this preference could be explained as being rooted in twelve-tone theory, a more realistic explanation is that Forte's preference stems from the fact that hexachords make better

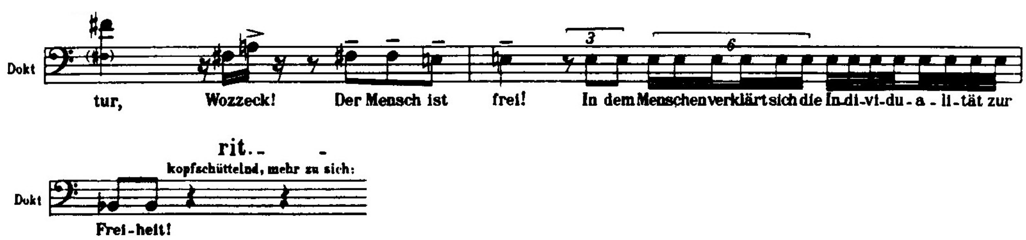
‘nexus’ sets, since they form K- and K_h-complexes that are neither too big nor too small.

Perle proved to be unimpressed, however, even with analyses based on z-related tetrachords, such as Janet Schmalfeldt’s observation (1983) that the two passages from Berg’s *Wozzeck*, shown in Example 1.1, involve tetrachords that are z-related.

Example 1.1: Two z-related tetrachords in *Wozzeck* (Schmalfeldt 1983, Exs. 1-2).

a. Act II/Scene 2, mm. 171-73 (one of the Doctor’s themes)

b. Act I/Scene 4, mm. 500-2



Despite the fact that he takes issue with Schmalfeldt’s segmentation in Example 1.1b, the main problem for Perle was a question of meaning. What does it mean to have the same interval vector? He writes, “I can discover these connections between Z-related collections only by subjecting them to an analytical scrutiny that has nothing whatever to do with my intuitive experience as a listener or as a composer” (1990, 168). To paraphrase, the tetrachords in the passages from *Wozzeck* are z-related, but so what? Perle’s argument is certainly at least partially true, since there is no clear

aural relationship between the two collections. Nevertheless, Perle planted the seeds for a response in his following statement: “nor can I find anything in the way the composer unfolds these respective collections that points up a formally significant special relation between them, such as one might find in reference to such a relation at the conclusion of a tonal piece” (1990, 169). The answer for Perle, it seems, is more analysis. The z-relation is only as relevant as is its application in music, where it can function as a compositional concept. Just because Perle did not see any analyses that convincingly demonstrate relationships between z-related sets does not mean there are no pieces in which the properties of z-related sets are exploited.

1.2 Other research developments on the z-relation

1.2.1 Z-related sets and the M5-relation: O’Connell’s Tone Spaces

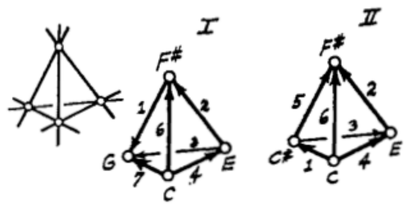
Already quite soon after the discovery of the z-relation music theorists observed links between it and set multiplication. One of the earliest accounts on set multiplication, in Walter O’Connell’s 1962 article in *Die Reihe*, made such a connection to the z-relation. In this article, O’Connell introduced a number of unique concepts based on mathematics and group theory, such as tone lattices and the pitch-class/order-position exchange.[17] His tone lattices demonstrate, among other things, that the all-interval tetrachords ([0137] and [0146]) are related by what he called the ‘chromatic-fifths transformation’ (41ff)—that is, M5 (multiplication by 5).

---

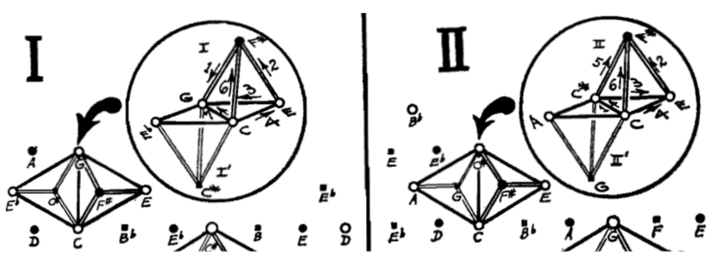
[17] Dave Headlam (2006) has written on the first part of the article, which concerns the op/pc exchange. He makes a few comments about O'Connell's treatment of the all-interval tetrachords, in the context of his earlier Berg analyses.

O'Connell defined the 'tone lattice' as a 3-dimensional pitch-class space that represents "the six different intervals by six different directions in space" (53). The tetrachord is the basic unit of the tone lattice, since the tetrahedron is the polygon that has six edges between the corners. Of the various tetrachords, O'Connell finds that there are only two tetrachords that have all six interval classes, namely the all-interval tetrachords, [0146] and [0137] (see Example 1.2).[18]

Example 1.2: The two tetrachords with all six interval classes (O'Connell 1962, 53).



Example 1.3: The two tone lattices, generated from the two all-interval tetrachords (O'Connell 1962, 55).



These two tetrahedrons lead to two different tone lattices, which he called the Diatonic-harmonic (I) and the Pentatonic-chromatic (II). Except that they are in three dimensions, O'Connell's tone lattices are very much like neo-Riemannian *Tonnetze*: they are fixed spaces upon which transformations may occur. Using the two

---

[18] O'Connell related his tone lattice to observations made in crystallography, indicating that it is an instance of what crystallographers call the "face-centered cube lattice" (1962, p. 56).

tetrahedrons as basic building blocks, tone lattices are generated by stacking together tetrahedrons of the same type so that the edges (intervals 1-6) of the tetrahedrons line up. O'Connell's next illustration demonstrates a segment of a lattice (see Example 1.3). Since, as O'Connell noted, the two generating tetrachords are M5-related, the same is true of the two lattices in the diagram. The two lattices are, in effect, different only in that the edges for intervals 1 and 5 are swapped and reversed in direction. Although the implications and uses of such spaces will not be considered here, O'Connell's tone lattices serve to demonstrate a fascinating early observance of the connection between the all-interval tetrachords and set multiplication.

1.2.2 Robert Morris and Set-Group Systems

In the article "Set Groups, Complementation, and Mappings among Pitch-Class Sets" (1982), Morris surveyed the relationship between the M5-relation and the z-relation in general, going beyond just the all-interval tetrachords. In this article he focused on general aspects of the groups of sets formed by the various equivalence classes. His work on set groups helps clarify the relationships between the different universes of set classes. As he showed, though some z-related sets are related by M5, like the all-interval tetrachords, not all of them are; and consequently, the two set-class universes based on interval vector and on T_nI/M5 overlap, but neither includes the other.

Using the twelve-tone operators (TTOs)—transposition, inversion and multiplication by 5—as well as the interval vector, Morris parsed the 4096 pitch-class

sets into four different groups of set classes.[19] Each group of set classes, based on the four equivalence relations, is called a 'set-group system', which are as follows: SG(1) is the group based on T_n equivalence; SG(2) is based on T_n and T_nI (the standard set group); SG(3) is based on T_n, T_nI and M5; and SG(v) is the group based on interval vector equivalence (the group adopted in Forte 1964). These four set groups all have a different number of set classes, as shown in Example 1.4.[20] Since the number of set classes differs between the four set groups, it is thus possible that a pair of sets could be equivalent under SG(3), for example, but not SG(1), such as the pitch-class sets {0,1,3,5} and {0,2,4,5}.

Example 1.4: The number of set classes in the four set groups.

Cardinality	#pc-sets	SG(1)	SG(2)	SG(v)	SG(3)
0	1	1	1	1	1
1	12	1	1	1	1
2	66	6	6	6	5
3	220	19	12	12	9
4	495	43	29	28	21
5	792	66	38	35	25
6	924	80	50	35	34
7	792	66	38	35	25
8	495	43	29	28	21
9	220	19	12	12	9
10	66	6	6	6	5
11	12	1	1	1	1
12	1	1	1	1	1
Total	4096	352	224	201	158

[19] Later, Morris (1987) abandoned inversion and defined the TTOs as only T_n and multiplication by ±1 or ±5. Hook 2007b discusses the mathematics that underlie these cardinality numbers, and cites Riordan 1958 as an example source of the formulas.
[20] The table in the example compiles information from Morris 1982, Table 10 (p. 126) and the table in Morris 1987, p. 80.

As Morris showed, the three set groups based on the TTOs—SG(1), SG(2) and SG(3)—can be hierarchized, so that the latter set groups include the former, as in:

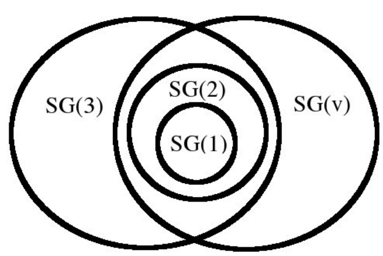
$$\mathrm{SG}(1) \subseteq \mathrm{SG}(2) \subseteq \mathrm{SG}(3)$$

That is to say, if a pair of sets are equivalent in SG(n), then the pair will necessarily be equivalent in SG(m) for all $m < n$. This relationship is to be expected, of course, since all three set groups have transposition, and SG(2) and SG(3) have inversion; that is, the equivalence relations accumulate from SG(1) to SG(3). Like SG(3), SG(v) also includes SG(2); that is,

$$\mathrm{SG}(1) \subseteq \mathrm{SG}(2) \subseteq \mathrm{SG}(v)$$

SG(v), however, does not include SG(3), or vice versa (see Example 1.5). Though SG(v) and SG(3) overlap, neither one includes the other, and thus SG(v) and SG(3) cannot be placed into a hierarchy. Therefore, a pair of sets that are equivalent in SG(v) are not necessarily equivalent in SG(3), and vice versa. For example, the sets of z-related pair [01356] and [01247] are M5-related to themselves, and not to each other, while the two set classes [012] and [027] are not z-related, but are M5-related.

Example 1.5: Venn diagram of the hierarchy of the set groups.



Ultimately, the disagreement between SG(v) and SG(3) stems from the fact that not all z-related pairs (in SG(2)) have sets that are M5-related to each other. If all the

pairs did have M5-related sets, then SG(3) would in fact hierarchically include SG(v); but instead, there are 13 z-related pairs with sets that are not M5-related. Morris calls the sets of these latter 13 pairs “true” z-related sets since they cannot be resolved within SG(3).

1.2.3: Robert Morris: CUP and ZC-related sets

The discussion on SG in Morris 1982 only broadly described the z-relation, and thus did not reveal any particular properties of z-related sets. Using other methods, however, Morris uncovered many of the key features of z-related sets. In the work that followed his 1982 article (particularly Morris 1987, 1990 and 2001), Morris showed for the first time (in music theory) that z-related sets share particular subsets, and that some of the subsets are cyclic collections (or unions of cyclic collections)—key aspects of the z-relation that are central to the analytical methodology that I develop in Chapters 2-3.

In his 1987 *Composition with Pitch-Classes: A Theory of Compositional Design*, Morris outlined certain sets (some of which are z-related sets) that possess what he considered to be “special abstract features.” The cases he gives have what he would later call the *complement-union property* (CUP, the set-class property where all non-overlapping unions of two or more certain set classes yields the same set class or set classes), the first case being the all-interval tetrachords. He observed (p. 88) that 1) both all-interval tetrachords partition into the set classes [06] and [03], 2) that any non-overlapping combination of [06] and [03] yields one of the two tetrachords, and 3) that there is a non-overlapping union of the two tetrachords that forms the

'octatonic' collection, set class [0134679t]. Another case involved the all-trichord hexachord ([012478]), which as Morris showed, results from any non-overlapping union of [0167] and [04]. As will be discussed later in this chapter, these particular observations about the all-interval tetrachords and the all-trichord hexachord have had a strong impact on recent studies on the music of Elliott Carter, explored by writers such as Guy Capuzzo (1999 and 2004), and Adrian Childs (2005).

In addition to these particular observations, Morris revealed certain aspects of the z-relation through his investigations on the ZC-relation. His observations on the ZC-relation provide some of the earliest insights into the nature and structure of z-related sets, even though his work on the ZC-relation was not aimed at explaining the z-relation per se. Unlike the z-relation, which is a relationship shared by two sets of the same cardinality, the ZC-relation is a relationship between a set and its complement. In this regard, the ZC-relation can be seen as being an offshoot of Lewin's generalized hexachord theorem. The definition of the ZC-relation is as follows (Morris 1990, 178):

> DEF 3.4: Two pc-sets are said to be ZC-related if they are mutual complements and neither one can map into the complement of the other under a TTO.

(Note that, unlike above, in this definition the TTOs are transposition and inversion, and not M5.) Naturally, all z-related hexachordal pairs are ZC-related; however, besides the hexachords, there is also one pentachordal/heptachordal pair that is ZC-related: the set [01356] and its complement. Among the 46 z-related sets, there are only 7 pairs that are not ZC-related: [0146], [0137], [01247], [01457], [01258],

[01348] and [03458], and their respective complements. Though while all ZC-related sets are also z-related sets, the opposite is not true.

Though the ZC-relation is actually discussed in a number of Morris's works (1982, 1987 and 1990, among others), it received most attention in his 1990 article. In that article, Morris explored set-class complementation, sketching out some of the relationships between a pair of sets that together form the aggregate. Having already defined the ZC-relation in previous writings, here Morris sought to provide more detailed explanations for the ZC-relation. Since all but one of the pairs of ZC-related sets are hexachords (the other is the pair with 5-Z12 [01356] and its complement, 7-Z12), a majority of his theory centers around the hexachords.

To explain which hexachords are ZC-related, Morris provided four criteria for ZC-related hexachords. Though each hexachordal ZC-related pair may satisfy several of the criteria, each pair satisfies at least one of the criteria, which are as follows (Morris 1990, 204-205):

> Criterion 1: If hexachord Y includes X, a member of SC 5-12[01356], then Y and Y^{-} are ZC-related. Since X is ZC-related to X^{-}, and, by THEOREM 1.2, $Y^{-} \subset X^{-}$, it follows that Y and Y^{-} are ZC-related, since FY (for any TTO F) cannot be included in Y^{-} and also include X. Example: {01356} ⊂ {012356}; {012356} ∈ SC 6-3; {4789AB} ∈ SC 6-36.
>
> Criterion 2: Hexachords X and X^{-} are ZC-related if X = (Y ∪ Z) and X^{-} = (T_4Z ∪ T_8Z) where Y ∈ SC 3-12[048], Z is a trichord that contains no ic4s and Z, Y, T_4Y, and T_8Y partition the aggregate. The ZC-relation stems from the fact that, although X contains the same number of ic4s as X-, only X contains an "augmented chord," that is, a complete T_4 (or T_8) cycle. Example: {235} = Y; {048235} ∈ 6-39; {679AB1} ∈ 6-10.

> Criterion 3: Hexachords X and X^{-} are ZC-related if X = (Y ∪ Z), and $X^{-} = (T_3Z \cup T_6Z \cup T_9Z)$ where Y ∈ 4-28[0369], Z is a dyad that is not an ic3 or ic6, and Y, Z, T_3Z, T_6Z, and T_9Z partition the aggregate. A ZC-relation occurs because X contains a "diminished-seventh chord" (or complete T_3-cycle) and X^{-} does not. This kind of ZC-relation can be equivalently described as between X and X^{-}, where X consists of the nonintersecting union of a member of 4-28 and an ic n, where n is not 3 or 6 and X^{-} is the union of two diminished trichords (SC 3-10[036]) at Tn. Example: {3A} = Y; X = {3A258B}; X^{-} = {619407} (= {147690}).

> Criterion 4: Two ZC-related hexachords may be related under the T_nM and/or T_nMI operations. Where this is not the case, a ZC-pair maps to a different ZC-pair, or each member of the ZC-pair is invariant under these operations involving M. Like the two previous criteria, this involves T_n cycles as well, since M and MI map the T_1 or T_{11} cycles into the T_5 or T_7 cycles.

The first criterion is based on subset inclusion, and criteria 2-4 are based on interval cycles: criterion 2 is based on the 4-cycle, criterion 3 on the 3-cycle, and criterion 4 is based on the 1- or 5-cycles. As Morris noted, the first criterion—which states that if a hexachord is a superset of [01356], then the hexachord will be ZC-related to its complement—is neither generative nor generalizable; rather, it is a description that is limited to only the hexachords in mod12. There are four hexachords that satisfy the first criterion. Criteria 2 and 3 are similar to each other, both stating that complemental hexachords are ZC-related if one hexachord contains [048] (or [0369]), while the complementary hexachord is the union of multiple instances of the set that is the remainder once the [048] (or [0369]) is extracted from the first hexachord. There are nine complementary hexachordal pairs that satisfy either criterion 2 or 3, or

both. The fourth criterion involves the M5-relation. Unlike the other three criteria, criterion 4 does not weed out cases of non-ZC-related hexachordal pairs; nevertheless, Morris uses it as a catchall for the three remaining ZC-related hexachordal pairs that were not covered by the first three criteria.

One of his tables (Table 6 in Morris 1990) lists the 15 hexachordal pairs, and shows the corresponding criteria, as well as the various mappings under M5 (see Example 1.6). This table is particularly illustrative since it not only shows which criteria each pair satisfies, but also how the set classes behave under multiplication—the left-hand column in the table shows the various mappings under M5.

Example 1.6: The ZC-related hexachords, M5 mappings and the four criteria (Morris 1990, table 6, 206).

M/MI*	ZC-pair	criterion 4 $T_{1/5}$-cyc	criterion 2 T_4-cyc	criterion 3 T_3-cyc	criterion 1 5–12
M	6–3/36				x
M	6–4/37		x		
E	6–6/38	x			
M	6–10/39		x		
E	6–11/40	x			
S	6–12/41				x
M	6–13/42			x	
S	6–17/43		x		
E	6–19/44	x			
S	6–23/45			x	
M	6–24/46		x		
M	6–25/47				x
M	6–26/48		x		
S	6–28/49		x	x	x
M	6–29/50			x	

As the table indicates, there are three mappings: a ZC-related hexachord is 1) M5-related to itself (S), 2) M5-related to its complement (E), or 3) the ZC-related pair is M5-related to another ZC-related pair (M). Morris's observation about ZC-related sets and multiplication is, in fact, generalizable for all z-related sets: a pair (or group)

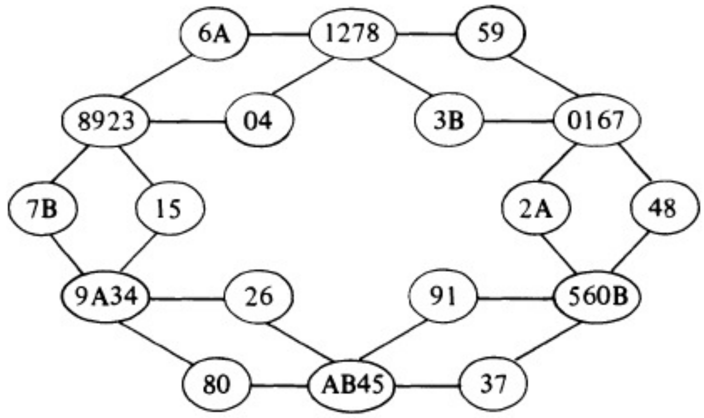
of z-related sets in $\mathbb{Z}_m$ (modulo *m*) always maps in one of these three ways for any M_x where *x* is coprime with *m*.

As with criteria 2 and 3, Morris finds that several ZC-related hexachordal pairs derive from subset unions. This discovery arose as an extension of his complement-union property (CUP), which is defined in the following (Morris 1990, 182):

> DEF 6.1: Given pc-sets V, S, and T, such that S ∈ SC(X), T ∈ SC(Y), and V ∈ SC(Z), if S ∩ T = { }, and S ∪ T = V, for all V, S, and T, then SC(Z) has the complement-union property.

A set class has CUP, in other words, if it can be formed by all possible non-overlapping intersections of two given set classes. As mentioned before, an example of CUP can be seen with the set [012478]—often referred to as the 'all-trichord hexachord' (ATH) since it contains forms of all twelve trichordal set classes as a subset—which has CUP since it is formed by any non-overlapping intersection of set classes [0167] and [04]. Since all intersections of the two sets form the ATH, it is possible, as Morris demonstrates, to create complement-union 'chains' (or wreaths), where each of the various transpositions of the two subsets are laid out as nodes. An example of such a chain is shown as Example 1.7, using the set classes [0167] and [04] to form various instances of the ATH. In the chain, all instances of two connected nodes form an ATH.

Example 1.7: The complement-union chain for the ATH, set class [012478] (Morris 1990, 188).



Such a chain does not explain how the sets of a ZC-related pair interrelate, since it involves only one set class. However, Morris extends the notion to ZC-related hexachordal pairs with a method for deriving ZC-related hexachords from 'ZC-partitions'. Like with criteria 2 and 3, deriving ZC-related hexachords from ZC-partitions is based on the idea that ZC-related hexachords share certain common subsets, and that one of the subsets is a cyclic collection.

ZC-partitions divide the aggregate into four subsets (3+3+3+3 or 4+4+2+2), where either one or two subsets are a cyclic collection and the other are of the same set class. For example, one possible partition of the aggregate divides the twelve pitch classes into the cyclic collections {0,6} and {2,8} (Y and Z), plus two 4-note sets of the same set class (W and X). There are four possible partitions with this arrangement (plus their transpositions and inversions by T_6, I_2 and I_8), which are shown in Example 1.8. For each of the four aggregate partitions, any union of either W or X

and Y or Z yields the ZC-related hexachord classes shown in the right-hand column of the example.

Example 1.8: ZC-partitions involving {0,6} and {2,8} plus two sets of the same set class (based on Morris 1990, table 7a, 212).

W	X	Y	Z	
1345	79AB	06	28	6-Z3/6-Z36
B134	579A	06	28	6-Z11/6-Z40
B237	59A1	06	28	6-Z19/6-Z44
479B	A135	06	28	6-Z25/6-Z47

Some ZC-partitions produce complementary hexachords of the same set class. However, as Morris explained, though not all ZC-partitions necessarily yield ZC-related hexachords, it is true that all ZC-related hexachords can be formed by ZC-partitions. ZC-partitions therefore cannot be used to enumerate or define the ZC-hexachords. Nevertheless, they describe the ZC-hexachords and provide some information about them—most importantly, ZC-partitions reveal that cyclic collections, in particular [06], play a key role in generating the ZC-hexachords. Though Morris asserted that other cyclic collections could generate ZC-hexachords, the examples that Morris gave that generate the 15 ZC-related hexachordal pairs are all based on some variant of the [06] cyclic collection—[06], [0167], [0268] and/or [0369]. Though in some cases the two cyclic collections are not the same—for

example, letting Y=[0268] and Z=[0369]—they are always based on the [06] cycle. Morris's partitions also underscore that the two sets of a ZC-related hexachordal pair are both comprised of cyclic collections plus some other collection of the same set class. However, since Morris's work is confined to the ZC-relation, it does not describe all of the z-related sets. Despite this, because he recognized that ZC-related hexachords share certain similar, if not identical, subset partitions involving cyclic collections, it can be said that Morris, through his work on the ZC-relation, was the first music theorist to identify some of the key properties of the z-relation.

1.2.4 Elliott Carter and the all-interval tetrachords

In the last few years, some music theorists have adopted Morris's theory to describe pitch relations in the music of Elliott Carter involving the all-interval tetrachords and the all-trichord hexachord.[21] The work in this area has uncovered certain key features of the z-relation in general, even though it has specifically focused primarily on only one z-related pair, the all-interval tetrachords (AITs). Until Clifton Callender and Rachel Hall's work on the z-relation (see Chapter 2), the recent theory on Carter represented the cutting-edge of the z-relation theory, even though it was not advertised as such.

Elliott Carter's compositional style shifted dramatically around 1980 and became much more centered around using certain core elements held in common amongst a body of works—as David Schiff (1988, 2) describes in the following:

---

[21] See, for example, Capuzzo 1999 and 2004, Childs 2005 and Jenkins 2010.

> In his middle years Carter felt compelled to exhaust a musical vocabulary with each composition. Since 1980, he has based a series of widely different works on similar premises: after years of plowing rocky soil, it was now time for the harvest.

In his dissertation (1999), Capuzzo outlined some of these premises held among the post-1980 works, focusing in particular on the pitch domain. Capuzzo recognized (as did Schiff) that many works prominently feature the all-interval tetrachords and all-trichord hexachords, and that these sets play a central role in Carter's method of pitch generation. In many of Carter's later works, the compositional design often involves an interweaving of various instances of particular set classes (usually the all-interval tetrachords and/or the all-trichord hexachord), and/or twelve-tone rows to predetermine the registers in which particular pitch classes appear.[22] Capuzzo (1999, 4-9) describes this overall compositional attitude as "variety within unity," in that certain source elements (such as the all-interval tetrachords) unfold to create ever-changing pitch structures.

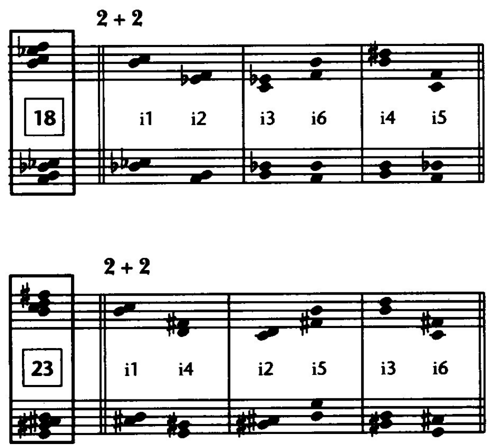
To weave together various instances of a predetermined set, it is imperative to know which subsets are shared between the various instances of the sets. Carter's 2002 book, *Harmony Book*, explores this topic thoroughly, and carefully charts out the various subsets of all of the sets in the twelve-tone universe. Concerning the all-

---

[22] Many twelve-tone rows used by Carter are what have come to be known as 'Link chords', as so named by Schiff (1998) since John Link had sent Carter in 1992 a list of all the all-interval twelve-tone rows that contains one or two instances of the all-trichord hexachord. See http://www.johnlinkmusic.com/LinkChords.pdf (accessed 12/03/2011), for a letter of explanation by John Link along with the list of rows.

interval tetrachords in particular, Carter recognizes that the only common dyadic partition shared by the two tetrachords is the two-part partition, [03]+[06].

Example 1.9: Comparison of dyadic partitions of AITs (Carter 2002, 364).



In Example 1.9, Carter shows for each set (sets #18 and #23, which correspond to 4-Z15 and 4-Z29) both the 'normal form' in the upper stave, and the inversion in the lower stave.[23] Each "measure" shows a 2+2 partition. The set [0146] (Carter's set #18) partitions into three possible dyad pairs: i1 and i2, i3 and i6, or i4 and i5. The set [0137] (Carter's set #23) also partitions into the dyad pair i3 and i6, but the other two partitions are different.

To explain the relationships between all-interval tetrachords in Carter's music based on the [03]+[06] partition, Capuzzo (1999) adopted Morris's notion of CUP, which shows that any non-overlapping combination of [03] and [06] dyads yields an all-interval tetrachord. In a later article (2004), Capuzzo expanded the idea of CUP and developed a series of transformations that change an all-interval tetrachord to a z-

---

[23] Carter's numbering system for the set classes is distinct from Forte's and Rahn's.

partner that shares one of the two dyads. These transformations are able to account for many of the relationships between all-interval tetrachords, benefiting from the fact that they touch on key aspects of the z-relation on the whole. Whether he recognized it or not, some of these transformations that are based on the [03]+[06] partition are representative of transformations that can be applied to many other z-related sets besides the AITs, and correspond quite closely to observations made by crystallographers on this very z-related pair. The transformations that I develop in Chapters 3 and 4, as we shall see, are very similar to Capuzzo's transformations.

In response to Capuzzo's work, however, Adrian Childs (2005) argued that in Carter's music all-interval tetrachords do not always overlap [03] or [06] dyads, but in fact often overlap one of the other intervals ([01], [02], [04] or [05]). Accordingly, Childs developed another series of AIT-specific transformations that handle pitch overlaps between all-interval tetrachords with any interval. Though while Childs may be correct to argue that in Elliott Carter's music the all-interval tetrachords interact in a variety of ways, for which the corresponding theory should be accountable, I find that Capuzzo's approach, which uses the [03]+[06] partition of the all-interval tetrachords, to be advantageous, since 1) it theoretically corresponds to the structure of the z-relation, and is thus generalizable for many z-related pairs, and 2) it is theoretically simple, involving one two or three transformation types, while being complete, in that it relates all 48 all-interval tetrachords.

Since Capuzzo's transformations are very similar to the ones that I develop in Chapters 2 and 3, I have decided to postpone a discussion on his transformations until

after mine are fully established. Thus, we will revisit Capuzzo's work in Chapter 4, with a comparison of his and my analyses of one particular passage from the music of Carter.

1.2.5 Soderberg and dual inversion

In his 1995 article "Z-Related Sets as Dual Inversion," Stephen Soderberg developed a theory to explain z-related sets. While he claims his motivation is driven by a compositional impulse, there is in the article a strong undercurrent of a desire to explain the mathematical origins of the z-relation. Much of his work is generalized to any modulus (not just mod12), and several of his examples explore these other microtonal universes. His work, overall, provides a method to derive z-related sets from scratch using a space, the 'Q-grid,' as a master superset from which z-related sets are selected. The z-related sets that are extracted from the Q-grid are related (due to structure of the Q-grid) by 'dual inversion'—a compound inversion that inverts each of the pitch classes by one of two different indexes. Though Soderberg's work on the z-relation was a break-through in its time, some of the theoretical apparatus have proven to be cumbersome and impractical due to the complexity of the Q-grid itself. Nevertheless, the Q-grid accurately demonstrated for the first time in music theory that z-related sets (at least some of them) are related by a transformation that affects two subsets in two different ways, an idea that will be very important in the chapters to come, as it relates to what I describe in Chapters 2 and 3 as 'partial' transposition or inversion.

Like Morris, Soderberg showed that z-related sets are based on unions involving cyclic collections. The approach that Soderberg took, however, is somewhat different in that Soderberg's Q-grids replaced Morris's aggregate. That is, instead of deriving z-related pairs by partitions of the aggregate, z-related sets are selected from Q-grids, which are the unions of cyclic collections.[24] Moreover, the process of deriving pairs of z-related sets from the Q-grid does not require that all of the pitch classes in the Q-grid are selected, thus adding a level of flexibility not found in Morris's explanations.[25]

The article (Soderberg 1995) begins by identifying seven "basic vector theorems," referring to two different vectors presented in his section 1: the interval-class vector and the spanning vector. The interval-class vector, which is equivalent to Forte's interval vector, is defined as the following (Soderberg 1995, 78):

> For any set $A \subseteq Z_m$, the "interval-class vector" of A is
> $V(A) = [a_1, a_2, \ldots, a_n]$
> where a_i = the number of pairs of elements in A which form the interval class i and … n is the largest interger $\leq m/2$. In other words, V(A) is an ordered n-ad representing the total interval-class content of A.

The spanning vector, by contrast, counts the interval classes that lie between the elements of two different sets. In this regard, the spanning vector can be seen as being equivalent to Lewin's IFUNC, except that the spanning vector counts only interval classes (1-6, excluding 0). The spanning vector is defined as the following (78):

[24] Morris 2002 moved beyond the aggregate and explored the partitions of double-aggregate multisets that generate z-related sets. See Chapter 2 for more on multisets.
[25] Morris 1982-3 did discuss the possibility of deriving combinatorial sets independently of the aggregate.

> For A, B ⊆ Z_m, the "spanning vector" V(A/B) is defined as for V(A), except that a_i is the number of pairs of elements (e, f) which form the interval class i where e ∈ A and f ∈ B.

The spanning vector does not count common tones (with interval class 0), and thus the spanning vector has the same number of entries as the interval-class vector. The reason for Soderberg's choice for not counting common tones, so it appears, is that using the 6-digit spanning vector allows for the possibility of developing an algebra of vector addition and subtraction that involves both the interval vector and the spanning vector, which we shall explore shortly.

As Soderberg explains, the spanning vector can be best understood as being the result of a multiplication table, where A is listed vertically on the side of the table, and B horizontally on the top. However, since the table tallies the interval classes, the intervals from the usual subtraction process are converted into interval classes 0 through *m*/2 (or (*m*-1)/2 if *m* is odd). One example multiplication table that Soderberg gives is shown as Example 1.10, which finds the spanning vector for the sets A={0, 1, 9, 12} and B={0, 9, 12, 13} in $\mathbb{Z}_{14}$ (integers mod14).

Example 1.10: Finding the spanning vector with a multiplication table for two sets in $\mathbb{Z}_{14}$ (Soderberg 1995, 79).

	0	9	12	13
0	0	5	2	1
1	1	6	3	2
9	5	0	3	4
12	2	3	0	1

Spanning vector:

[3331210]

The table lists the various interval classes 0-7 (mod14), and the spanning vector tallies the number of appearances in the table of each of the interval classes 1-7. Here we see that the three instances of ic0 in the table, which indicate three common tones, are not counted in the spanning vector.

After defining the two vectors, Soderberg gives seven basic vector theorems (Soderberg 1995, 81-82). These theorems define the interval vector and spanning vector, and define the vectors of sets that are the union of two or more sets. The seven theorems are as follows (proofs omitted):

2.1 $V(A/B) = V(B/A)$
2.2 $V(A) = V(T_xA) = V(I_yA)$
2.3 $V(A/B) = V(T_xA/T_xB) = V(I_yA/I_yB)$
2.4 $V(A/I_xB) = V(I_xA/B)$
2.5 $V(A/T_xB) = V(T_{-x}A/B)$
2.6 If $A \cap B = Ø$,
$V(A \cup B) = V(A) + V(B) + V(A/B)$
2.7 If $A \cap B \cap C = Ø$,
$V(A/(B \cup C) = V(A/B) + V(A/C)$

The first three theorems (2.1–2.3) cover the basics of the vector definitions, and need no commentary. The next two theorems (2.4–2.5) extend theorem 2.1, showing how transposition and inversion can be injected into 2.1. Theorem 2.6 defines the interval class vector for the union of sets A and B. Though Soderberg does not show it, the theorem can also be reoriented so that it defines the spanning vector, as in:

$$V(A/B) = V(A \cup B) - V(A) - V(B)$$

Lastly, theorem 2.7 defines the spanning vector between a set A and the union of two other sets. It should be noted that the 'if' clauses in theorems 2.6 and 2.7, which state

that A and B (and C) do not intersect, are necessary only to prevent multisets—sets with multiple entries, such as the set {0, 0, 1, 4}. Otherwise, if multisets are allowed, then the 'if' clauses are not needed and the theorems remain true.

Soderberg recognized that there is a relationship between the z-relation and cyclic collections—sets such as {0, 6} and {0, 4, 8} that are formed by a repeating generative interval. However, rather than showing how cyclic collections appear as subsets of z-related pairs (at least explicitly, as did Morris), Soderberg uses cyclic collections (Φ) to form Q-grids—spaces upon which z-related sets can map onto one another given certain preconditions. Soderberg defines the cyclic collection Φ as the following (82):

> Given $\phi|m$, $0 \le \alpha \le m-1$, then
> $$\Phi = \{e \in \mathbb{Z}_m : e = \alpha + \beta\phi\}$$
> where β ranges through $0,1,...,(m/\phi)-1$.

Φ is the cyclic collection that is generated by the interval ϕ. The variable α is the offset from 0. Thus, if $\phi = 3$ and $\alpha = 1$, for example, then Φ would be the pitch-class set {1, 4, 7, 10}.

Though Soderberg's definition of the cyclic collection is not an unusual one for pitch-class theory, the symbols that he chose, Φ and ϕ, strangely conflict with the standard use of those symbols in mathematics; in fact, his usage is in direct contradiction to the symbols' standard usage. In mathematics ϕ is generally used to designate a coprime of a given number. Soderberg, instead, uses it to designate a divisor. Also, the set Φ is often used in mathematics to designate the set of all coprimes ϕ of a number *n* that are less than *n*; instead, Soderberg uses it to signify a

cyclic collection. Despite these conflicts, I find that Soderberg's usage of the symbols is perfectly adequate in the music theoretical context, and that adopting Soderberg's symbols, as I do in Chapter 3, provides a level of coherence between his theory and mine.

The theorems concerning the set Φ serve as the basis upon which his Q-grids are developed. Like Morris, Soderberg noted that the two all-interval tetrachords form the octatonic collection under non-overlapping unions. He showed that this case is just but one case of a more general phenomenon, however, and that several z-related pairs can be extrapolated from H-sets—sets that are the union of two Φ-sets, as in the following (87):

$$H = (\Phi) \cup (T_x\Phi) = (\Phi) \cup (I_y\Phi)$$

Examples of H-sets include any set that is the union of two instances of a cyclic collection, such as the octatonic and hexatonic collections. To demonstrate the relationship between H-sets and the z-relation, Soderberg begins by showing that each H-set has what he calls a 'characteristic vector', defined as:

$$X(H) = V(\{e\}/H) \text{ where } e \in H.$$

That is, the characteristic vector is the vector of interval classes between one member of H and all the other members of H. For example, the characteristic vector of the octatonic collection is [112111], while for the hexatonic collection {0, 1, 4, 5, 8, 9} it is [101210]. Given that the cardinality of a set A is represented as <A>, Soderberg's theorem 3.18d generalizes Morris's observation about the all-interval tetrachords and the octatonic in the following (87):

If $A \cup B = H$, $A \cap B = \varnothing$,

$$V(A) - V(B) = \tfrac{1}{2}(\langle A\rangle - \langle B\rangle)X(H)$$

That is, if A and B form H under a non-overlapping union, then the difference of their vectors is half the difference of their cardinalities multiplied by the characteristic vector of H—for vector multiplication, each integer in the vector is multiplied by a given integer. If A and B are of the same cardinality, then it follows that ½(<A> – <B>)X(H) = ½(0)X(H) = [000000], and therefore V(A) = V(B). In other words, any partition that divides an H-set into two sets of equal cardinality will yield two sets that have the same interval-class vector, and thus that are either of the same set class or z-related. As Soderberg notes, if Φ is the aggregate, then this formula is the Hexachord Theorem. As I show in Chapter 2, the reverse statement of what the formula indicates—that any two z-related pairs can form an H-set under some union—would be true if H-sets were not limited to the union of two and only two Φ-sets, a limitation that appears to be imposed solely so that H-sets are able to form Q-grids. This reverse assertion will become clearer with the help of the Fourier transform (see Chapter 2).

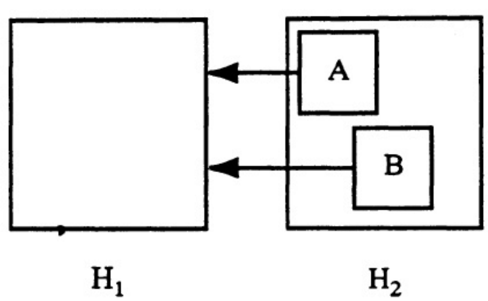
Though H-sets can produce z-related sets, Soderberg takes one more step, forming Q-grids from H-sets, in order to develop the notion of dual inversion. The transition from H-sets to Q-grids begins with a theorem on the spanning vectors between two sets (A, B) and an H-set (Theorem 3.20, p. 88):

Given $H_1 = (\Phi) \cup (I_x\Phi)$ and $H_2 = (T_y\Phi) \cup (I_xT_y\Phi)$,

$$\text{if } A, B \subseteq H_2,\ \langle A\rangle = \langle B\rangle, \text{ then } V(A/H_1) = V(B/H_1)$$

This formula uses two H-sets: one from which the two sets A and B are drawn, and one from which the intervals between it and the sets A and B are counted. As the formula shows, if the sets A and B are both in H_2 and are of the same cardinality, then the spanning vector between A and H_1 will be the same as the spanning vector for B and H_1. Soderberg illustrates this notion with the figure shown as Example 1.11. What is notable here is that A and B do not have to be of the same set class. For example, given $\Phi = \{0,4,8\}$ in $\mathbb{Z}_{12}$, let $H_1 = (\Phi) \cup (I_5\Phi) = \{0, 1, 4, 5, 8, 9\}$ and $H_2 = (T_3\Phi) \cup (I_5T_3\Phi) = \{2, 3, 6, 7, 10, 11\}$, two hexatonic collections. Then, the sets A = $\{3, 6, 7\}$ and B = $\{2, 6, 10\}$, which are both chosen from H_2 and are of different set classes, share the same spanning vectors to H_1: $V(A/H_1) = V(B/H_1) = [0303630]$. In all cases, in fact, the spanning vector is the characteristic vector of H_1 multiplied by the cardinality of the sets ($[0303630] = 3 * [0101210]$) .

Example 1.11: Equivalent spanning vectors from A and B to H_1 (Soderberg 1995, 89).



The next step on the way from H-sets to Q-grids is to pull the sets A and B from two different H-sets. This happens by defining two new H-sets, H_1' and H_2', which expand the original pair of H-sets to a group of four. As Soderberg

demonstrated, the new arrangement, which pulls A from H_2, and B from H_2', is able to generate z-related sets. His theorem 3.23 states:

Given:

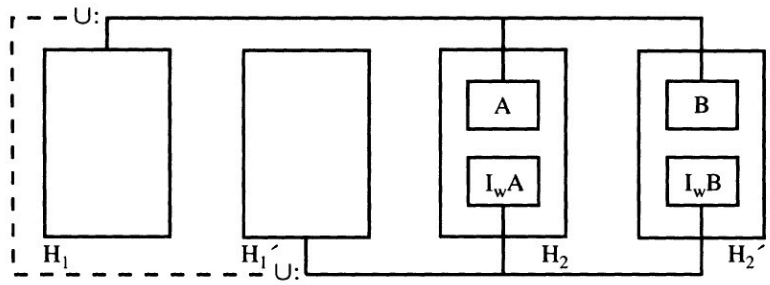
$H_1 = (\Phi) \cup (I_w\Phi)$
$H_1' = (I_x\Phi) \cup (I_wI_x\Phi)$
$H_2 = (T_y\Phi) \cup (I_wT_y\Phi)$
$H_2' = (I_xT_y\Phi) \cup (I_wI_xT_y\Phi)$

with $w\text{-}x = k\phi/2$. For any $A \subseteq H_2$ and $B \subseteq H_2'$ with $\langle A\rangle = \langle B\rangle$,

$V((A\cup B) \cup H_1) = V(I_w(A\cup B) \cup H_1')$

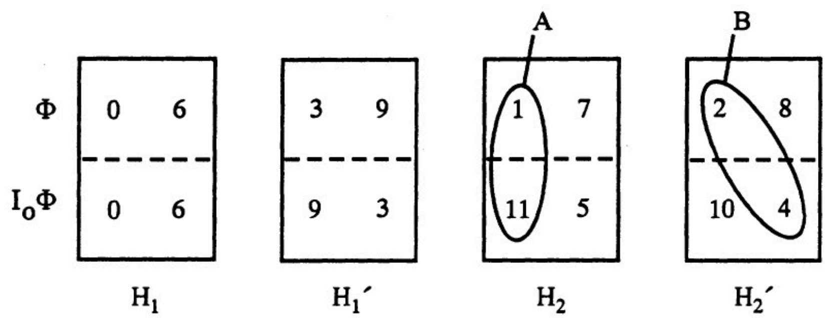
The diagram representing the theorem is given as Example 1.12. The theorem shows that the two sets $(A\cup B) \cup H_1$ and $I_w(A\cup B) \cup H_1'$, marked by the two solid lines in Example 1.12, share the same interval vector. Here we begin to see Soderberg's notion of dual inversion for the first time: $A\cup B$ inverts at I_w onto $I_w(A\cup B)$, while H_1 inverts at I_x onto H_1'. The latter inversion may not be so clear, since there is a sort of 'twist' in the inversion: that is, though from H_1 to H_1' the subsets Φ and $I_x\Phi$ are related by I_x, and $I_w\Phi$ and $I_wI_x\Phi$ are related by I_{-x}, the condition that $w–x = k\phi/2$ (where k is some positive or negative integer) has the result that H_1 and H_1' are always of the same set class and are related by I_x. The reason for this is that if $w–x$ is some multiple of $\phi/2$, then $I_wI_x\Phi = I_xI_w\Phi$ (they are commutative). For example, if $\Phi = \{0, 6\}$, $w = 1$ and $x = 4$ in mod12, then H_1 and H_1' would be $H_1 = \{0, 6\}\cup\{1, 7\} = \{0, 1, 6, 7\}$ and $H_1' = \{4, 10\}\cup\{9, 3\} = \{3, 4, 9, 10\}$, which are related by I_4 (i.e. I_x).

Example 1.12: Two sets, $A \cup B \cup H_1$ and $I_w(A \cup B) \cup H_1'$, that have identical interval vectors (Soderberg 1995, 90).



Soderberg gives two examples where z-related sets are derived from such a formation of H-sets. Let us consider here his second example, where $\phi = 6$, $\Phi = \{0, 6\}$, $w = 0$, $x = 3$ and $y = 1$ (mod12). His figure 11b gives the H-set formation for these variables (see Example 1.13). The sets A and B (circled in the figure) are two equal-cardinality sets that were arbitrarily chosen. By applying the theorem, we find that the sets S and T have the same interval vector, where $S = A \cup B \cup H_1 = \{0, 1, 2, 4, 6, 11\}$ and $T = I_w(A \cup B) \cup H_1' = \{8, 9, 10, 11, 1, 3\}$. These two sets share the same interval-class vector ([342231]), and since they are not of the same set class (a fact determined on our own) they must be z-related. H-set formations like the one shown here can derive a large number of z-related sets, though Soderberg does not indicate exactly how many.

Example 1.13: Example H-set formation, from which a z-related pair is derived (Soderberg 1995, 92).

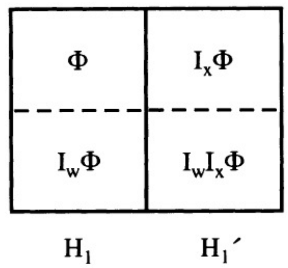


Though in this example the sets H_1 and H_1' are technically multisets, Soderberg treats them as regular sets; that is, both H_1 to H_1' are both 2-note sets, and not 4-note sets. Nevertheless, the z-relation remains true in both cases, whether or not they are considered as multisets. Furthermore, though in Soderberg's example the sets A and B are of the same set class, they need not be the same for the theorem to yield z-related sets. That should be clear, though, since the sets A and B act as a single set, transforming together by the same inversion operation (I_w).

The final step, moving from H-set formations to Q-grids, is a relatively small step, but it adds another level of generality. Instead of just being able to select sets A and B, the Q-grid allows for multiple sets to be selected, albeit with certain restrictions. The benefit is that now one single Q-grid can produce (in theory) more z-related pairs. On the Q-grid, the union of the chosen sets (Q) transforms onto another set with the same interval-class vector (Q*) through Q-inversion (i.e. dual inversion). Together the two sets Q and Q* are called a Q-pair—i.e. a pair of sets with the same interval vector related by dual inversion. Like the formations of H-sets above, the Q-

grid is an array of sets based on Φ, laid out as a series of cells; but for the Q-grid each cell is equivalent to the union of H and H′ above, as illustrated in Example 1.14.

Example 1.14: Basic cell of the Q-grid (Soderberg 1995, 93).



The formal definition of a Q-grid is (Soderberg 1995, 95):[26]

$$[m, \phi; w, x] = \bigcup_{\alpha} \begin{bmatrix} \Phi & \mathrm{I}_x\Phi \\ \mathrm{I}_w\Phi & \mathrm{I}_w\mathrm{I}_x\Phi \end{bmatrix}$$

An example can be seen with the following Q-grid, where ϕ = 12 and Φ = {0} (from Soderberg 1995, 96):

$$[12,12;0,6] \qquad \left[\begin{array}{c|c} 0 & 6 \\ 0 & 6 \end{array}\right] \left[\begin{array}{c|c} 1 & 5 \\ e & 7 \end{array}\right] \left[\begin{array}{c|c} 2 & 4 \\ t & 8 \end{array}\right] \left[\begin{array}{c|c} 3 & 3 \\ 9 & 9 \end{array}\right]$$

Each square in the Q-grid is a basic cell, as shown in Example 1.14 (here, Φ is the single pitch class 0). The variable *y*—the transposition level between the Φ-sets of adjacent cells—is left out of the formal definition, but in the accompanying prose Soderberg suggests that *y* is assumed to be 1 in all cases, since the selection process

[26] In the transition from H-sets to Q-grids, Soderberg tacitly switches around the variables x, y and w, replacing w with x, x with y, and y with a new variable α that has a slightly different meaning. A side effect of this is that the sets H_1 and H_1' appear to rotate from a vertical position to a horizontal position, so that H_1 sits above H_1', rather than to the left of it. Since there is no actual rotation involved, only a change in variables, I personally find this to be confusing and unnecessary. For this reason, I have modified his formula to use the variables as defined previously.

allows for cells to be skipped. The variable α is the total number of cells, and when used as subscript it indicates the cell number (where $\alpha = \{0,1,\ldots,\alpha-1\}$). Each cell of the Q-grid is the union of H_α and H_α'. Soderberg never explained what α ideally should be, and why there are four cells in the example just given. Though Q-grids could go on indefinitely, theoretically having an infinitely large number of cells, it seems that Soderberg chose to stop adding cells once all (or most) of the twelve pitch classes appeared in the cells, thus avoiding the issue of generating multisets. Presumably, then, when such redundancies begin to appear, one can simply just choose whether or not to stop adding cells.

Once the Q-grid has been formed, then we can create a z-related pair by choosing a set Q, which is done simply by selecting pitch classes from the cells in the Q-grid, and then by deriving the corresponding Q*. There are three options when choosing pitch classes from a cell: the choice set P_α of pitch classes can be 1) either H_α or H_α'; 2) the union of A and B, where $A \subseteq H_\alpha$, $B \subseteq H_\alpha'$ and A and B are of the same cardinality; or 3) the null set, with no pitch classes selected. The set Q, then, is the union of all choice sets P_α.

The set Q* is generated by applying Q-inversion (dual inversion) to the set Q. This is achieved by considering the selection sets P_α chosen from each of the cells, and inverting each of the selection sets about a certain axis depending on which type of selection set was chosen from the three options above. These are as follows:

1) if P_α is either H_α or H_α', then P_α^* is $I_x(P_a)$.
2) if $P_\alpha = (A \cup B)$, then P_α^* is $I_w(P_a)$.
3) if $P_\alpha = \varnothing$, then $P_\alpha^* = \varnothing$.

In other words, each of the pitch classes in Q map onto a pitch class in Q* by one of two inversions. The two sets are thus related by inverting one subset at I_x, and the other at I_w.

For an example Q-pair, let us consider one of Soderberg's examples that uses the Q-grid shown above, where w = 0, and x = 6. Using the Q-grid, certain pitch classes are selected from the Q-grid using the three options for each cell (see Example 1.15). After the pitch classes are chosen, Q-inversion finds the sets P_α*, which together form Q*. In this case, the chosen selection sets and their Q-inverted partners together form the all-interval tetrachords.

Example 1.15: Example usage of a Q-grid, which derives the all-interval tetrachords (Soderberg 1995, 96, circles added).

$$\begin{bmatrix} 0 & 6 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ e & 7 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ t & 8 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 9 & 9 \end{bmatrix}$$

$P_0 = \{0, 0\} = \{0\}$	$P_0{*} = I_6P_0 = \{6\}$
$P_1 = \{1, 7\}$	$P_1{*} = I_0P_1 = \{5, 11\}$
$P_2 = Ø$	$P_2{*} = Ø$
$P_3 = \{3, 3\} = \{3\}$	$P_3{*} = I_0P_3 = \{9\}$
$Q = \{0, 1, 3, 7\}$	$Q{*} = \{5, 6, 9, 11\}$

In the example, Soderberg has again reduced out the pitch class duplications, avoiding the multiset. However, as noted before, the interval vectors of Q and Q* are the same whether or not the sets are considered as multisets. Furthermore, it should be clear by now that the selection sets should not all be of the same type (options 1-3), or else the sets Q and Q* will simply be related by inversion, either by I_x or I_w. In other

words, dual inversion (and not just a regular single inversion) must take place for Q and Q* to be z-related sets.

With the Q-grid, Soderberg demonstrated that z-related sets potentially map onto each other under dual inversion, a transformation that inverts two common subsets of the two z-related sets at two different indexes. In addition to this discovery, the Q-grid provides a method for deriving unknown z-related pairs in other moduli besides mod12, which did not exist in music theory before this article.

1.2.6 Some observations and criticisms of Soderberg's Q-grids.

Soderberg's Q-grids successfully describe many z-related sets, and deliver for the first time a strategy for building z-related pairs from scratch. The theory is robust since Q-grids can generate (in theory, at least) an infinite number of z-related sets. However, of the 43 pairs of z-related sets in mod12, there are 4 pairs that cannot be generated by Q-grids. Interestingly, though, the four pairs of z-related pairs that Q-grids do not explain, as Soderberg observes, happen to be the same four pairs that that fall under Morris's 4th criterion of ZC-related pairs: the fours pairs where one set of the pair has an [0369] subset. Nevertheless, the number of z-related sets that Q-grids can produce is astounding.

In addition to generating the z-related sets, Q-grids demonstrate that z-related sets can transform onto each other under dual inversion. This observation is completely original, extending well beyond Morris's aggregate-based explanations, which promote more of a static view of how z-related sets (or ZC-related sets) interrelate. Soderberg's description that z-related sets can map onto each other by

affecting pitch classes in more than one way is the first instance of a transformational perspective applied to the z-relation, a line that continues into research into Carter (with Capuzzo and Childs), as well as my own research.

Despite its high explanatory power, the theory of Q-grids has some significant disadvantages. First, and perhaps most importantly, not all Q-grids produce z-related pairs, and the ones that do produce z-related pairs also produce sets of the same set class—a problem that Morris also faced with his ZC-partitions. In order for the Q-grid to be used as a generator of z-related sets, some sort of filtering would have to be applied to weed out the non-z-related cases. Soderberg offers only a few suggestions to alleviate this problem, but his suggestions help to filter out only some of the non-z-related cases. Furthermore, since some Q-grids do not yield z-related pairs, one needs some method for determining whether or not a given Q-grid will generate any z-related pairs at all. However, it appears that the only way that this can be done is by considering the structure of the z-related pairs that would be created. In other words, the resultant z-related pairs need to be known before one can know if the Q-grid will generate z-related sets, which ultimately defeats the purpose of using Q-grids to derive z-related sets.

As for the analytical applications of Q-grids, take, for example, some questions that might arise while either composing or analyzing music with z-related sets. First, given the set $\{0, 2, 3, 8, 9, 16\}$ in $\mathbb{Z}_{24}$, is the set a z-related set, and if so, to which set(s) is it z-related? To answer this question using Q-grids, one would have to figure out which Q-grids contain this set, and try all the cases to see if there is a Q-

inversion that generates a set of a different set class. Such a process would be enormously challenging, especially without any rules to help figure out which Q-grids contain particular sets.

Another question might be, given a pair of z-related sets, what transformations can be applied to transform one z-related set to the other? Q-grids give a partial answer to this question with dual inversion. With the example above, we saw that I_6 of the set {0} and I_0 of the set {1, 3, 7} yields a set, {5, 6, 9, 11}, that is z-related to {0, 1, 3, 7}; but how might one generalize the two inversions so that they are applicable to other transpositions and/or inversions of {0, 1, 3, 7}? Furthermore, are there other possible transformations between z-related sets besides dual inversion, and if so, how do the various transformations relate to each other? Unfortunately, the theory of Q-grids is limited only to dual inversions.

Despite these limitations, however, Soderberg's theory is indeed illuminating. The reasons that Q-grids work stems from the fact that Q-grids are based on some of the fundamental properties of z-related sets: particularly, that z-related sets are based on cyclic collections, and that z-related sets can transform onto one another by transforming two (or more) common subsets. Though dual inversion, as we shall see, is not always necessary, being that often a single inversion or transposition is sufficient, the idea comes surprisingly close to some of the formulas drafted by crystallographers, as outlined by Callender and Hall.

2. Recent Advances Concerning the Z-relation

In order to introduce a transformational perspective of the z-relation, this chapter explores some of the recent advances of z-relation theory. Through the media of crystallography concepts, I aim to clarify some of the basic tenets on which the z-relation is based, and, with the aid of information provided by crystallographers, to show that numerous z-related sets (among the various moduli) can be described as algebraic expressions involving subset unions of cyclic collections. In this chapter I also discuss the Fourier transform—a mathematical formula, which was introduced into pitch-class set theory by David Lewin and Ian Quinn, that measures the "weight" of a set in respect to the cycles—and use it to show the extent to which cyclic collections participate in the z-relation. The magnitude portion of the results of a Fourier transform is closely related to the interval vector, yet the Fourier transform also provides additional information (concerning phase) beyond that given by the interval vector. All of this information will later be synthesized (in Chapter 3) into an original general theory of z-related pitch-class sets, introducing a series of generalized pitch-class-set transformations that change a z-related set to its z-partner—what I call *z-transformations*.

2.1 Crystallography versus music theory

Music theorists now have an unusual opportunity to draw on another field to further expand some aspects of pitch-class set theory. As only just realized, music theorists were not the first to write about concepts such as T_nI set classes and the z-

relation; rather, crystallographers (those who study atomic structures) preceded music theorists by some twenty years in outlining the *cyclotomic* sets (which are analogous to the set classes in music theory) and *homometry* (which is analogous to the z-relation).[1] Despite the differences in aim between music theory and crystallography, the two fields managed to concurrently develop set theories that are very much related; but because of the differences in aim and technique, crystallographers have explored certain areas of set theory that music theorists have not, and vice versa. One area that has received special attention from crystallographers, however, is in fact the topic of homometric sets (i.e. z-related sets).[2]

In crystallography, homometric sets arise as a problem when determining atomic structures. The main method of determining atomic structures begins by finding the 'intervals' between the atoms of a molecule (the interatomic distances), and interpreting these intervals to determine the atomic structure. Some possible atomic structures, however, are "not identical nor…mirror-images of one another", and yet "give identical sets of interatomic distances" (Patterson 1939, 940). In such cases, it is impossible to reconstruct the atomic structure from the interatomic distances alone since there exists more than one possible structure with those interatomic distances (i.e. there are homometric structures). Accordingly,

---

[1] O'Connell 1962 already recognized a connection between the set theories of music theory and crystallography, though he did not demonstrate how they relate. Callender and Hall 2008, O'Rourke, et al. 2008, Senechal 2008 and Ballinger, et al. 2009 have all explicitly disscussed the connection between music theory and crystallography.

[2] Among the many works by crystallographers on homometric sets, some of the principal ones are Patterson 1939 and 1944, Bagchi and Hosemann 1954, Bullough 1961 and 1964, Buerger 1959 and 1975, Rosenblatt and Seymour 1984, and Rosenblatt 1990.

crystallographers have explored homometry to be able to recognize such anomalous cases, and to be able to determine in such cases which of several possibilities is the correct atomic structure. In short, their success in measuring atomic structures depends in part on uncovering the underlying principles of homometry.

Though music theorists have made some of the same observations concerning the z-relation as have crystallographers about homometry (such as the existence of z-related sets in other moduli, and that cyclic collections are involved), overall the theories given by crystallographers about the phenomenon surpass any of the music theoretical explanations, both in terms of breadth and precision. For instance, crystallographers have shown that homometric sets appear in various dimensions, as well as in both periodic and linear spaces. They have shown that the homometric sets can be classified into various types, and that there are a number of methods to represent homometric sets mathematically. Their work makes apparent that the mod12 z-related sets are only a small sample of very broad phenomenon (homometry) that manifests in many different spaces, and that some of the homometric sets (including the mod12 z-related sets) can be generalized and expressed as algebraic equations.

Further to some of the technical explanations concerning homometry, the work in crystallography on homometry also provides an alternate perspective to the traditional conceptions of the z-relation in music theory. Unlike most music theorists (not including those mentioned in Chapter 1), who have generally treated the z-relation as a circumstantial byproduct produced by the conventions of pitch-class set

theory—sets that share identical interval content—crystallographers describe homometry pragmatically as a mathematical puzzle that has solutions. They have devised formulas to explain homometric sets, showing that certain z-related sets have particular common subsets arranged differently. These formulas invite a transformational perspective of the z-relation, in that they show that a z-related set transforms onto it partner when certain subsets are transposed or inverted.

In music theory, the z-relation has generally been characterized as a perceptual problem—that is, in order to be musically relevant, the z-relation should express something about the acoustic character of musical pitch-class sets. As we saw before, George Perle (and to some degree John Clough) used precisely this argument against the z-relation. Such a line of reasoning is completely understandable in music studies: since it has yet to be shown what it is exactly that the z-relation reflects, the option of last resort has been to treat the z-relation as an expression of some underlying acoustic similarity. If not acoustic similarity, then what exactly does the z-relation show? Even the proponents of the z-relation have made connections between the z-relation and sound similarity. Joseph Straus (2000, p. 80), for instance, wrote, "[s]ets in the Z-relation will sound similar because they have the same interval-class content, but they won't be as closely related to each other as sets that are members of the same set class," adding further that z-related sets are like cousins in a family, whereas sets of the same set class are siblings.

In this context, however, the z-relation is problematic: the supposed high degree of similarity between z-related sets (suggested by the interval vector) is not

provable, and is, in my opinion, highly suspect. There are other pairs of pitch-class sets, for instance, that do not share the same interval vector, and that are arguably more acoustically similar than are z-related sets.[3] If the z-relation does not reveal something about acoustic similarity, then one begins to wonder how it is even significant as a musical concept. One may indeed conclude that the z-relation is an anomalous relationship that evolves from the presumed system of measurement (tallying up the interval classes), and that it has little bearing on music analysis.

In comparison to the problematic perspective commonly found in music theory, the perspective of the crystallographer on z-related sets is refreshing. Unlike with most music theories, in crystallography no particular interpretation is proposed as to what homometricity 'means' (like acoustic similarity), other than that it is a phenomenon that complicates the analysis of atomic structures, and that it appears to manifest under certain describable conditions. As crystallographers have shown, many z-related sets can be simply explained as deriving from particular combinations of subsets. An analyst coming from this perspective, then, would not place interest on the z-related sets themselves, but on the particular combinations of subsets. Accordingly, music analysis based on this perspective would aim to show how certain subsets produce z-related sets by way of a counterpoint using specific intervals, and that z-related sets transform onto their partners by rearranging the subsets. To get a

[3] For example, one could argue that the collections [0248] and [0268], which are both subsets of the whole-tone collection, are more acoustically similar than are the all-interval tetrachords.

better understanding of this perspective, less us now review some of the claims that crystallographers have made concerning homometric sets.

2.2 Homometry in crystallography

The discovery of homometry arose as a problem in x-ray diffraction analysis (or spectroscopic analysis), which is one of the primary methods used today for analyzing atomic structures. For this kind of analysis, x-rays are passed through a given material, producing a series of diffraction patterns. In order to determine the positions of the atoms in the atomic structure, the x-ray diffraction patterns are interpreted using the Patterson function (which is derived from the Fourier transform) to find the interatomic distances.[4] The atomic structure is deduced by representing the structures as sets with distances between the elements determined by the Patterson function. Usually, knowing the interatomic distances is enough for one to be able to successfully abstract the atomic structure; however, since x-ray diffraction patterns show interatomic distances, which are akin to intervals between pitch-classes, the Patterson function is unable to distinguish between homometric atomic structures. Such cases are a problem for crystallographers, since in those cases the data given by x-ray diffraction can only suggest that the atomic structure is one of several possible atomic structures.

As noted before, crystallographers preceded music theorists in both enumerating the set classes and discovering the z-relation. In 1944, Arthur Lindo

---

[4] Buerger (1959, ch. 1-2) provides a helpful history of the mathematics involved in x-ray spectroscopy, including an explanation of the Patterson function.

Patterson defined the *cyclotomic* set as the equivalence class based on rotation and reflection (equivalent to transposition and inversion), which is analogous to the set class in music theory. In one table, Patterson (1944, 196) listed the number of cyclotomic sets in mod8 through mod16 (see Example 2.1).[5] For each n (modulus) the table lists the number of cyclotomic sets (i.e. set classes) of cardinality r, showing for example that in mod12 there are 224 cyclotomic sets.[6]

Example 2.1: Patterson's table enumerating the cyclotomic and homometric sets (Patterson 1944, 196).

TABLE I. Survey of cyclotomic sets (r points on a period divided into n equal parts). The number of possible arrangements was obtained by first considering the number of r partitions of n. The number of cyclic arrangements (clock order neglected) of each r partition was then considered. In all, 2664 cases ($4<r<n/2$, $n\leq 16$) were investigated. 390 pairs, 7 triplets, and 3 quadruplets were found.

	Possible arrangements								Number of pairs				
n	$r=1$	2	3	4	5	6	7	8	4	5	6	7	8
8	1	4	5	8	5	4	1	1	1	—	—	—	—
9	1	4	7	10	10	7	4	1	0	—	—	—	—
10	1	5	8	16	16	16	8	5	0	3	—	—	—
11	1	5	10	20	26	26	20	10	0	0	—	—	—
12	1	6	12	29	38	50	38	29	1	3	15	—	—
13	1	6	14	35	57	76	76	57	1	0	1	—	—
14	1	7	16	47	79	126	133	126	0	6	6	48	—
15	1	7	19	56	111	185	232	232	0	5	23	9	—
16	1	8	21	72	147	280	375	440	2	10	31[a]	42[b]	183[c]

[a] 3 triplets are here counted as pairs. If all pairs of the triplets are counted, 6 pairs should be added.
[b] 4 triplets are here counted as pairs. If all pairs of the triplets are counted, 8 pairs should be added.
[c] 3 quadruplets are here included as far as their complementary pairs are concerned. If the four non-complementary pairs of each quadruplet are included 12 pairs should be added.

[5] Table reprinted in Buerger 1959, 44.
[6] Hook 2007b discusses the mathematics behind these familiar cardinality numbers (1, 6, 12, 29, 38, 50…) in mod12, and shows that they can be found using Polya's Enumeration theorem, published 1937.

Furthermore, homometric sets were also discovered long before the z-relation appeared in music theory. Linus Pauling and M. D. Shappell first made the discovery in 1930 when they noticed that the mineral bixbyite could have one of two possible atomic structures given the x-ray diffraction patterns. Patterson (1939) later labeled such sets as 'homometric'. Patterson's table, shown above, also shows in the right-hand column the "number of pairs" of homometric (z-related) sets, including as well the homometric triples and quadruples in mod16. By the time the Hexachord Theorem was observed by Babbitt and Lewin in the late 1950s, crystallographers had already described the set class and the z-relation, and had begun to postulate theories on how the z-relation works.

Since the atomic structure of many materials are laid out in a lattice, where a particular pattern is repeated continuously, crystallographers have generally focused on sets in periodic spaces. Furthermore, as opposed to the single-dimension periodic space in pitch-class set theory, the sets used to represent atomic structures are generally either in 2 or 3 dimensions. Nevertheless, in order generalize the theory, crystallographers have written about sets in various dimensions, including the 1st dimension. Because of this, crystallographers have often addressed 1st-dimension periodic spaces, such as the mod12 pitch-class space.

Several publications in crystallography on homometric sets begin by first considering the single-dimension case, and then continue to the cases in multiple

dimensions as to accommodate the analysis of atomic structures.[7] Rosenblatt and Seymour (1982) begin by considering a case in a linear (non-periodic) 1-dimensional space (equivalent to pitch space): given two real-number multisets[8]—U, $V \subset \mathbb{R}$—the multisets $U+V$ and $U–V$ (defined in Example 2.2) are homometric (NB: here, they define the sum of sets differently from the 'multiset sums' discussed below):

Example 2.2: Homometric pair in linear (non-modular) space (Rosenblatt and Seymour 1982, 343).

$$U + V = \{u + v: u \in U, v \in V\}$$

$$U - V = \{u - v: u \in U, v \in V\}$$

The two equations in Ex. 2.2 are analogous to Cartesian products of U and V, except that they either add together or subtract each pair of elements.[9] As an example, they give the sets {0, 1, 3, 8, 9, 11, 12, 13, 15} and {0, 1, 3, 4, 5, 7, 12, 13, 15}, formed from the sets $U = \{6, 7, 9\}$ and $V = \{–6, 2, 6\}$. Starting with the element {6} of U, for example, the set $U + V$ is made by summing the element with the three elements in V, making the set {0, 8, 12}, and continuing likewise with the other elements in U.

Though this basic formula is in non-periodic space—and thus does not account for the z-related sets in mod12—it shows a fundamental aspect about many z-related sets: they arise from two (or more) distinct unions involving addition and

[7] Buerger 1950 and 1959, Bullough 1961 and Rosenblatt 1984 have this particular trajectory.
[8] Traditionally in music theory we talk about sets. However, this formula, and those that I explore and develop in Chapters 2 and 3, requires that we expand our purview to include multisets—i.e. sets that potentially have multiple copies of elements.
[9] A Cartesian product of a pair of sets X and Y is the set of all ordered pairs whose first component is an element of X, and the second an element of Y.

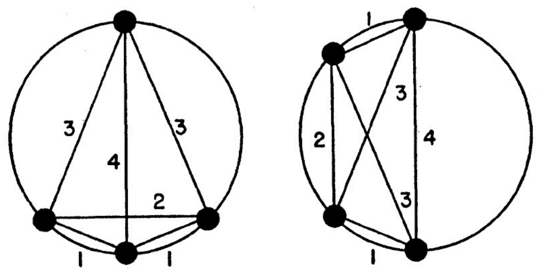
subtraction. However, as Rosenblatt and Seymour show, many (but not all) homometric sets arise from such formulas. The remainder of their article explores methods for analyzing cases where homometric sets cannot be described as set unions.

Besides the non-periodic case just given, some of the earliest examples given by crystallographers of homometric pairs in the 1st dimension are homometric pairs in a periodic space. Patterson 1944 begins by considering "a simple homometric pair," shown as Example 2.3, which is in mod8. After a discussion of cyclotomic sets (set classes), Patterson then generalizes the tetrachordal pair, describing it in terms of intervals based upon one undefined variable *a* (see Example 2.4).[10] The generalized version is instead in mod1, and has two fixed intervals that are not dependent on *a*—1/4 and 1/2.

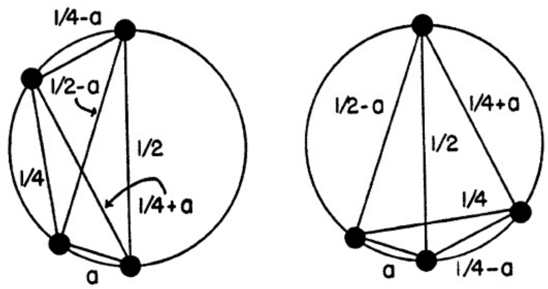
The generalization in Example 2.4 shows that there are an infinite number of such tetrachords, depending on the value of *a*, which can be any real number $0 < a < 1/4$. The pair in Example 2.3, for instance, derives from the general formula in Example 2.4, being the pair in mod8 where $a = 1$. Bullough 1961 also gives particular attention to this pair, which he considers to be "an interesting homometric pair" (263). Like Patterson, Bullough also writes the pair as an algebraic expression in mod1, but instead uses an expression that shows the elements of the sets (see Example 2.5).

[10] Patterson cites the mathematician Paul Erdös as a source of the generalization.

Example 2.3: Tetrachordal homometric pair (Patterson 1944, 197).



Example 2.4: Tetrachordal homometric pair generalized (Patterson 1944, 199).



Example 2.5: Formula for the tetrachordal homometric pair (Bullough 1961, 263).

$S_1 = (0, a, 1/4, 1/2{+}a)$

$S_2 = (0, a, 1/2{+}a, 3/4)$

When translated to mod12, the tetrachordal pair in Example 2.5 would be the following:

$S_1 = (0, a, 3, 6{+}a)$ mod12

$S_2 = (0, a, 6{+}a, 9)$ mod12

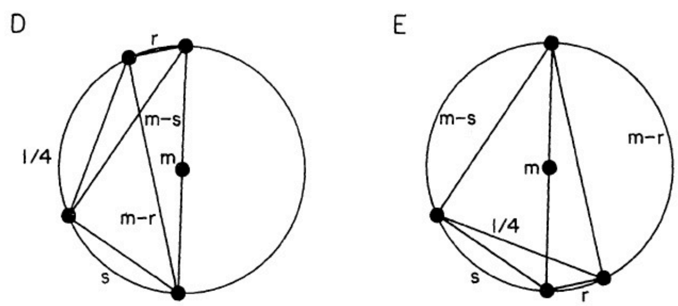
The pair in mod12 is representative of the all-interval tetrachords ([0146] and [0137]), so long as *a* is an integer not divisible by 3 (otherwise, the two sets are congruent). In its generalized version given by Patterson and Bullough, this pair is the only homometric pair with 4 elements that can be constructed algebraically. In this

regard, the pair is a special case. As proven later by Rosenblatt (1984, 336-7), the pair is one of only two possible tetrachordal homometric pairs (see Example 2.6). The other pair (Example 2.6/ii) is the tetrachordal pair in mod13, [0146] and [0237], which cannot be represented as algebraic equations involving unions of two or more subsets—that is, there is only one solution (and multiples thereof) for Example 2.6/ii.

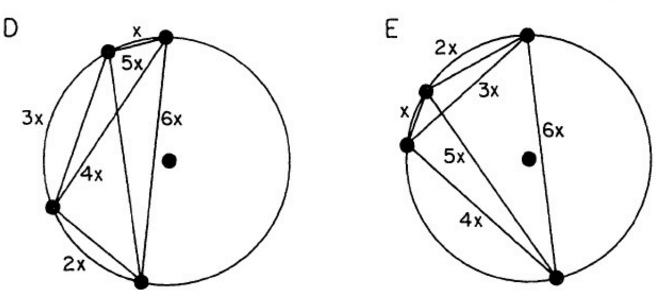
Besides the tetrachordal homometric pair shown as Example 2.4, Patterson showed that the algebraic approach can be applied to derive other homometric pairs. Alongside his graphic of the tetrachordal pair, he provides a further generalization that combines some cyclic collection plus 2 additional elements (Example 2.7).

Example 2.6: Only two possible homometric tetrachordal pairs (Rosenblatt 1984, 337).

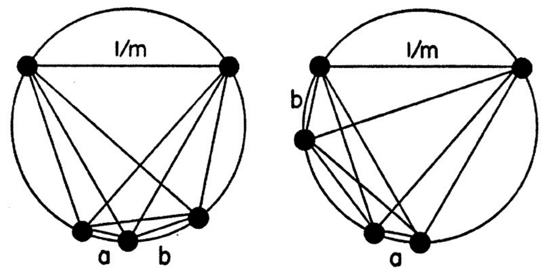
i) *Choose* r, $s \in (0/4)$ *with* $r+s=1/4$ *and let* $m=1/2$.



ii) *Let* $x=1/13$.



Example 2.7: Homometric pair with m+2 elements (Patterson 1944, 199).



The graph in Example 2.7 includes an equilateral triangle (the cyclic collection) plus a dyad. The triangle is to be understood as a cyclic collection of any cardinality. The complete period (mod1) is divided by m (not to be confused here with modulus) to make a cyclic collection with m elements. Two additional elements are situated so that the intervals a and b—the intervals between each of the two additional elements to the nearest element(s) in the cyclic collection—together add up to $1/(2m)$. The tetrachordal pair just mentioned derives from this formula, where $m = 2$ (with the tritone as the cyclic collection), as does the pentachordal z-related pair [01348] and [03458] in mod12, where $m = 3$ (with the augmented triad).

In addition to the examples already shown, Patterson also gives some examples of homometric triplets in mod16 of cardinalities 6 and 7. For these latter examples, however, Patterson does not provide mathematical derivations, but rather just shows the homometric triplets as examples of possible homometric groups.[11]

[11] In music theory, z-related triplets such as these were discovered almost forty years later by Lewin (1982).

The methodology presented thus far is the starting point for the present theory. Patterson's and Rosenblatt's formulas demonstrate that z-related sets are formed by subset unions that satisfy certain conditions.[12] However, in order to expand the methodology to include all of the z-related sets in mod12, I provide an alternate set of conditions. As I show, as long as the sets of the z-related pair share a common discrete subset partition—a partition that fractures a given set into more than one disjunct subsets—with at least one cyclic-collection subset (Soderberg's Φ, such as [06] and [048] in mod12), then the z-related pair is algebraically expressible.[13] However, before we begin to the analyze the details of such formulas (in Chapter 3), let us consider some more points that help to explain why only certain homometric sets can be described by algebraic formulas, and why cyclic collections are involved in such formulas. For the former, we will consider a particular distinction made by crystallographers that divides the homometric sets into various types; for the latter, we will explore the Fourier transform as a means for determining certain set properties.

[12] One discrepancy that arises between sets and multisets is in the definitions of 'set union', 'multiset union' and 'multiset sum'. Given the multisets {0, 1, 1, 2} and {0, 1, 6}, the union is {0, 1, 1, 2, 6} while the sum is {0, 1, 1, 1, 2, 6}—that is, the multiset sum (indicated by the sign $\uplus$) registers the total number of copies of each element between the two sets, while the multiset union registers the maximal number of copies among the two sets. I do not use 'multiset union', and thus all references of 'union' should be understood only as 'set union'. Most of the following formulas, however, use multiset sums. Though I sometimes resort to using 'union' for convenience, ultimately multiset sums are implied.

[13] I cannot prove this claim, but of the examples that I have looked at in moduli 8-30, I have yet to find a counterexample.

2.3 Psuedohomometric sets and homomorphs

As noted by Rosenblatt and Seymour, not all homometric sets can be represented algebraically. Some homometric sets have a structure that cannot be expressed as binary equations involving addition and subtraction; rather, these sets are instead 'indivisibly' homometric. The z-related set in mod13 periodic space mentioned before—the tetrachordal pair [0146] and [0237]—is one such case. Homometric sets of this kind cannot be expressed algebraically using common subsets; rather, they stand uniquely homometric, as each pair is the product of a unique pattern within the given space. They do not share any cyclic collections, and thus they cannot be expressed as algebraic equations.

Whether or not a z-related pair can be expressed algebraically is closely related to a distinction noted by Hosemann and Bagchi (1953). They observed that there are two kinds of homometric sets: pseudohomometric sets and homomorphs, which are defined formally as the following (238):

> "Homomorphs are characterized by the fact that they remain homometric under any suitable affine transformation"
>
> "Pseudohomometry…[h]omometric structures which under a suitable affine transformation degenerate to enantiomorphic or congruent structures."

Let us take a moment to parse out these statements. 'Congruence' is the equivalence class based on transposition, and 'enantiomorphy' is the equivalence class based on inversion (plus transposition). In a single-dimension space, the 'suitable affine transformations' are the three standard transformations: transposition, inversion and

multiplication—though multiplication by any integer is possible (not just 5 or 7 in mod12). However, since by definition homometric sets are not related by transposition or inversion, the 'suitable affine transformation' in question is just multiplication. That is to say, in a periodic pitch-class space pseudohomometric sets are the homometric sets that multiply by some multiplicand to the same set class (degenerate), and homomorphs are the z-related sets that remain homometric when multiplied by any integer.[14] All pairs of z-related sets in mod12 are pseudohomometric and degenerate to the same set class when multiplied, for example, by 6—both sets in each pair have the same division of pitch classes between the two whole-tone collections. On the other hand, the two sets of the z-related (homomorphic) pair in mod13, [0146] and [0237], remain homometric when multiplied by any integer. The latter pair never degenerates under multiplication to the same set class; thus, it is a homomorphic pair.

To get a better idea of what 'degeneration' entails, let us consider an example—for instance, the hexachordal pair in mod12, [012469] and [013468] (see Example 2.8). First, we multiply both sets by each integer from 2 to 6. There is no need to not consider 1 since it is redundant, and we can ignore the integers greater than half the modulus ($m/2$) since they duplicate the results of the first half. Of the five multiset pairs, two are z-related (multiplying by 3 and 5), while the other three are of the same set class. One of the two z-related pairs (multiplying by 3) has more than one copy of each element, while the other (multiplying by 5) is another z-related

[14] Only integers since fractions involve division, which is not allowed in modular math.

hexachordal pair that is different from the starting pair. The other three multiplicands yield instances of set degeneration. That is, multiplying the two z-related sets by 2, 4 and 6 yields two multisets of the same set class. Though this example pair has three instances of set degeneration, only one instance of set degeneration would have been sufficient to prevent the pair from being homomorphic.

Example 2.8: Set degeneration with the pair [012469] and [013468] in mod12.

[012469]	* 2	= [002468]	[013468]	* 2	= [002468]
	* 3	= [003366]		* 3	= [000369]
	* 4	= [000448]		* 4	= [000448]
	* 5	= [013457]		* 5	= [023458]
	* 6	= [000066]		* 6	= [000066]

The distinction between these two types of homometric sets rest on the fact that the sets of a homometric pair (or triplets, etc.) always share the same interval vector when multiplied by any integer, as they will always multiply to either homometric sets or sets of the same set class. If two sets share the same interval vector, then it is impossible to multiply them by the same integer so that they become sets with different interval vectors. Given a pair of homometric sets, then, multiplication can only 1) yield the same homometric pair (by either mapping the two sets onto each other or onto themselves), 2) yield another homometric pair, or 3) turn the pair into a pair of sets of the same set class. If one of the possible multiplicands produces sets of the same set class, then the homometric pair is pseudohomometric.

Though set degeneration does not correspond exactly to whether the sets of a homometric pair can be expressed algebraically, the two are closely related. For

instance, homomorphs reside exclusively in prime-number moduli where there are no cyclic collections to be shared—no partition with cyclic collections is possible, and thus it is evident that they cannot be described algebraically using cyclic collections.[15] Homomorphs sometimes share identical discrete partitions, but they never share a partition that includes a cyclic collection as one of the subsets.

However, the opposite is not true for the pseudohomometric sets: that is, pseudohomometric sets do not necessarily always partition into the same subsets when partitioned according to the cyclic-collection subsets. In other words, once all of the cyclic collections have been extracted from each set of a pair of pseudohomometric sets, it is not guaranteed that the two remaining subsets—that is, the residual non-cyclic-collection subset of each pseudohomometric set—will be of the same set class. Rather, there appears to be four ways that the pseudohomometric sets partition according to cyclic collections, which I define as four types of pseudohomometry:[16]

Type 1: A pair of pseudohomometric sets that share the same cyclic-collection subsets, and that have residual subsets of the same set class.

Type 2: A pair of pseudohomometric sets that share the same cyclic-collection subsets, and that have residual subsets that are homometric.

Type 3: A pair of pseudohomometric sets that share the same cyclic-collection subsets, and that have residual subsets that are neither of the same set class nor homometric.

---

[15] At this point, I know of no homomorphic pair that exists outside of the prime-number moduli (such as mod13, mod17, mod19, etc.).

[16] It should be noted that I have drawn these four types drawn empirically, and thus they may not be entirely conclusive.

Type 4: A pair of pseudohomometric sets that contain (and thus share) no cyclic-collection subsets.

Type 1 corresponds to the majority of homometric sets (including all the z-related sets in mod12). We have already seen examples of this type, such as the all-interval tetrachords, [0146] and [0137], which both partition into one [06] cyclic collection and a residual [03] non-cyclic collection.

Type 2 pseudohomometry is more rare. Pseudohomometric sets of this type share cyclic-collection subsets, but have residual non-cyclic-collection subsets that are homometric (specifically, Type 4 pseudohomometric). For example, there is a homometric pair in mod24 (among several pairs) where both sets partition into an instance of set class [0, 12] (a cyclic collection in mod24) plus one homometric pair (see Example 2.9; $\mathcal{Z}$ indicates that the two sets are homometric, and thus z-related).[17]

Example 2.9: Type 2 pseudohomometric pair (S_1 and S_2) in mod 24.

$S_1 = \{0, 1, 5, 6, 8, 12, 14\}$
$S_2 = \{0, 3, 4, 5, 9, 11, 17\}$
$S_1 \ \mathcal{Z} \ S_2$

$S_1 = T_1 \cup \{0, 12\}$
$S_2 = T_2 \cup \{5, 17\}$
$T_1 = \{1, 5, 6, 8, 14\}$ (sc[0, 4, 5, 7, 13])
$T_2 = \{0, 3, 4, 9, 11\}$ (sc[0, 2, 7, 8, 11])
$T_1 \ \mathcal{Z} \ T_2$

[17] There is no established symbol to indicate homometry. Bullough used the symbol)-(, while most other crystallographers (such as Rosenblatt) avoided ascribing a symbol at all. Later, Callender and Hall (2008) used the symbol $\mathcal{Z}$, referencing the z-relation from music theory. My own preference is to preserve the tradition from music theory, and to continue to use $\mathcal{Z}$.

Example 2.10: Homometric triple in mod16.

$U_1 = \{0, 1, 4, 5, 7, 8, 10\}$ => [0, 8] ∪ [0, 3, 4, 6, 9]
$U_2 = \{0, 1, 2, 4, 5, 8, 11\}$ => [0, 8] ∪ [0, 1, 3, 4, 10]
$U_3 = \{0, 3, 4, 5, 6, 9, 12\}$ => [0, 8] ∪ [0, 3, 4, 6, 9]

The third type of pseudohomometry is also rare. Sets of this type share set-class-equivalent cyclic-collection subsets, but have residual non-cyclic-collection subsets that are neither of the same set class nor homometric. Take, for example, the homometric triplet in mod16 shown in Example 2.10. Two of the three sets, sets U_1 and U_3, share the partition that extracts the [0, 8] cyclic collection, and are thus Type 1 pseudohomometric. However, the other set, set U_2, partitions into a [0, 8] cyclic collection plus a non-homometric set of a different set class; the set U_2 is thus Type 3 pseudohomometric with U_1 and U_3.

The fourth type of pseudohomometry differs from the other three types since sets of this type do not contain any cyclic-collection subsets. Sets of this type remain 'indivisibly' homometric like the homomorphs, such as sets T_1 and T_2 above: they possess no cyclic-collection subsets, and thus the two sets cannot share a partition based on cyclic collections. Among the Type 4 pseudohomometric sets, there are also several homometric pairs that are of a subtype (the subtype is unique to Type 4) where the span of the pitch classes in both sets is less than half of the modulus. For instance, take the pseudohomometric pair in mod24 shown as Example 2.11.

Example 2.11: Type 4 pseudohomometric pair in mod24.

$V_1 = \{0, 1, 2, 6, 8, 11\}$
$V_2 = \{0, 1, 6, 7, 9, 11\}$

In both sets, V_1 and V_2, the pitch classes are compressed to within one half of the modulus—seen by the fact that the interval between the first and last pitch classes is 11, which is less than 12. Such sets cannot possess a cyclic collection, and are thus 'indivisibly' homometric; however, a homometric pair of this subtype is also homometric in linear (non-periodic/pitch) space, as well as all of the other moduli greater than or equal to twice the span of the pitch classes. For example, the pair just given (V_1 and V_2) exists and is a homometric pair in all moduli ≥ 22, as well as in linear space. The sets T_1 and T_2 from above are Type 4 pseudohomometric, but they are not of this subtype, since one of the sets (T_1) does not fit within one half of the period. Table 2.1 lists a few homometric sets of this subtype. The pairs in the table are all pseudohomometric; but as mentioned, they are all Type 4 pseudohomometric sets and thus cannot be explained in terms of subset unions, and they are all of the subtype where the span of the pitch classes (in normal order) is less than half of the modulus.

Table 2.1: Homometric pairs that are also homometric in non-periodic pitch space, as well as all moduli greater than or equal to *x*.

Homometric pair		**Exists in moduli ≥ *x*, and in linear space**
{0, 1, 2, 6, 8, 11}	{0, 1, 6, 7, 9, 11}	22
{0, 1, 2, 3, 4, 6, 7, 8, 11}	{0, 1, 4, 5, 6, 7, 8, 9, 11}	22
{0, 1, 3, 6, 7, 8, 12}	{0, 1, 5, 6, 7, 9, 12}	24
{0, 1, 2, 5, 7, 9, 12}	{0, 1, 5, 7, 8, 10, 12}	24
{0, 1, 2, 3, 5, 6, 7, 9, 12}	{0, 1, 3, 4, 5, 6, 7, 10, 12}	24
{0, 1, 2, 3, 4, 6, 8, 9, 12}	{0, 1, 4, 6, 7, 8, 9, 10, 12}	24
{0, 1, 3, 4, 5, 7, 9, 10, 12}	{0, 1, 3, 5, 6, 8, 9, 10, 12}	24
{0, 1, 2, 3, 5, 7, 8, 11, 12}	{0, 1, 2, 4, 5, 6, 9, 11, 12}	24

It would certainly be interesting to explore these z-related sets in non-periodic pitch space. However, in general, I focus only on Type 1 pseudohomometric sets, since only these sets can be represented using Patterson's methodology of expressing sets in terms of algebraic equations. However, since the Type 1 pseudohomometric sets include the z-related sets in mod12, the subset-union methodology is sufficient for handling most of the z-related sets that have import in a music-theoretical context. As for the homometric sets that cannot be expressed as algebraic expressions (the homomorphs and some of the pseudohomometric sets), Rosenblatt 1984 demonstrated these sets can be explained by representing them as polynomials (such as $x^0 + x^1 + x^4 + x^6$). Accordingly, two sets are homometric if there exists some other polynomial (the *spectral unit*) that multiplies one homometric polynomial into the other (324).

> If D, E ∈ K[G], then D and E are homometric if and only if there exists a spectral unit U ∈ K[G] such that U * D = E.

Due to the complexity involved, and the lack of strong relevance for music analysis, I will not pursue any further explanations for the sets that cannot be constructed algebraically. Instead, the focus will be to provide explanations for the Type 1 pseudohomometric sets in modular space that can be explained with basic algebra alone. For now, however, we will set aside the algebraic formulas, and continue with an explanation as to why pseudohomometric sets share partitions involving cyclic collections, which can be found using the Fourier transform.

2.4 The Fourier transform

The Fourier transform is a function that decomposes a waveform into its constituent partials. In crystallography, the Fourier transform is used (in the form of the Patterson function) to interpret the data from x-ray diffraction analysis, and thus forms an integral part of how crystallographers deduce atomic structures. Though in music theory the Fourier transform has traditionally been used to analyze the distribution of partials in waveforms (with spectral analysis or audio signal processing), it can also be used to analyze pitch-class sets, as shown by David Lewin (1959 and 2001) and Ian Quinn (2004 and 2007). The possible interpretations that can be wrought from the information provided by the Fourier transform are too many to list here, and such a task would be beyond the scope of this study. Nevertheless, the Fourier transform and the interval vector show similar information, and thus we can use them interchangeably to define the z-relation. The Fourier transform, however, provides some additional insights not shown by the interval vector about the nature of the z-relation, such as insights on the role of the cyclic collection in the z-relation.

The following section provides a brief account of the explanations of the Fourier transform by David Lewin and Ian Quinn. After showing how their work reveals aspects about the cyclic-content of z-related sets, I extend their explanations to incorporate another component of the Fourier transform (vector angles) to propose an alternative explanation for z-related sets. Overall, in this section I show that z-related sets (in all known cases) share the partition has the most pitch classes in

cyclic-collection subsets, and that any two z-related sets (in any modulus) can be positioned so that their union is a set that is itself a union of cyclic collections.

2.4.1 Lewin and FOURPROP(*x*)

Among the many contributions of Lewin's first published work (1959) is the use of the Fourier transform for the analysis of pitch-class sets. In this article, as well as in his later 2001 article that revived some of the theory from the earlier article, the complex mathematics of the Fourier transform are translated into a series of set properties to address particular musical problems. Lewin asks (p. 299):

> Suppose we have collections P, Q, and R. Suppose the interval function between P and R is the same as the interval function between P and Q (i. e. for every i between 0 and 11 inclusive, the number of intervals i between notes of P and notes of R is the same as the number of intervals i between notes of P and notes of Q). Can we conclude that Q is the same collection as R?

Though the question is not necessarily related to the issue of the z-relation, Lewin's novel solution certainly is. He found the answer to the question to be yes, given certain restrictions. To explain his answer, he introduced an interpretation of the Fourier transform, interpreting it as five Fourier properties (FOURPROP) that any set may or may not possess, which are summarized as the following (Lewin 2001, 5-6, deriving from Lewin 1959, 299-300):[18]

> Pcset X has FOURPROP(1) if X can be expressed as a disjoint union of tritone sets and/or augmented-triad sets. (exceptional property)

[18] The FOURPROP(x) labels were introduced in Lewin 2001, while the original 1959 description used the more conventional labels (whole-tone property, etc.). The ordering shown here is the reverse of Lewin's ordering.

> Pcset X has FOURPROP(2) if, for any (0167)-set K, X has the same number of notes in common with T3(K), as it has in common with K. (tritone property)
> Pcset X has FOURPROP(3) if, for any augmented-triad set A, X has the same number of notes in common with T6(A), as it has in common with A. (augmented-triad property)
> Pcset X has FOURPROP(4) if X has the same number of notes in common with each of the three diminished-seventh-chord sets. (diminished-seventh-chord property)
> Pcset X has FOURPROP(6) if X has the same number of notes in one whole-tone set, as it has in the other. (whole-tone-scale property)

The five properties test whether or not a given set is the union of cyclic collections (FOURPROP(1)), or whether the pitch classes of the set are evenly distributed among certain cyclic or cyclic-based collections (FOURPROP(2-6)). A given pitch-class set may or may not possess any (or all) of the five properties, and in this sense the properties are Boolean tests for pitch-class sets: either a set has FOURPROP(*x*) or it does not. The set {0,1,6,7}, for example, has FOURPROP(6), since it has the same number of pitch classes in one whole-tone collection as it does in the other, as well as FOURPROP(1), since the set is a composite of two tritones; but since the set {0, 1, 6, 7} does not satisfy any of the other conditions, it enjoys (to use Lewin's word) only two of the five properties.

The Fourier properties are based on the divisors of 12 (1, 2, 3, 4 and 6). FOURPROP(5) is excluded since it is identical to FOURPROP(1), and therefore redundant. The number 12 is technically also a divisor of 12, but Lewin excludes FOURPROP(12) as well. Though FOURPROP(12) is theoretically possible, it would

be the property that tests whether or not a set has any pitch classes—the only set with FOURPROP(12) being the null set.

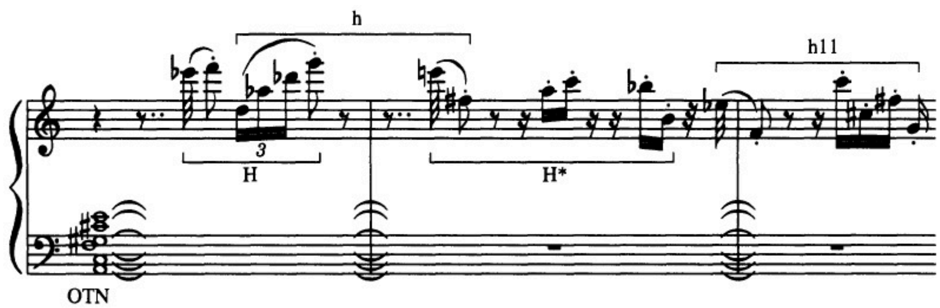
Using these five properties, Lewin posited an answer to his question in the following (Lewin 1959, 300):

> If P, Q, and R are collections of notes, and if both Q and R enjoy at least the same of the above five properties as does P, and if the interval function between P and R is the same as the interval function between P and Q; then Q and R are identically the same collection.

The result is that if two sets possess at least the same Fourier properties as a third set, then set-class identity between the two sets (Q and R) can be determined solely by the interval functions from P to those sets. Overall, the find is a subtle one, but it reveals a certain correspondence between the Fourier transform and the behavior of the interval function. However, due to the specificity of Lewin's question and solution, Lewin's Fourier properties remained dormant until 2001, when he revived them to answer a new question. In his 2001 article, Lewin asks: given that a 'constant-IFUNC pair' is a pair of sets that have an interval function where all 12 ordered intervals appear the same number of times (with a vector like [xxxxxxxxxxxx]), "[w]hat are [the] necessary and sufficient conditions on a pair of (non-empty) sets <X,Y>, that they should be a constant-IFUNC pair?" (Lewin 2001, 6). The question is asked in reference to a hypothetical passage of music, shown as Example 2.12. The four hexachords bracketed in the upper staff—the set H, the inversion h, the complement H*, and the transposition h11—each form a constant-IFUNC pair with the 'Ode to Napoleon' hexachord (also known as the hexatonic set) in the lower staff. The

interval function from H (and H*, etc.) to OTN, and from OTN to H, is [333333333333].

Example 2.12: Hypothetical passage with constant-IFUNC pair (Lewin 2001, 2).



Again, the answer to the proposed question involves the five Fourier properties, however this time the answer is that the two sets must together possess all five of Fourier properties, where each property is held by one or both sets. In the example shown above, the OTN has FOURPROP(x) for x = 1, 2, 4 and 6, while H has FOURPROP(x) for x = 3, together covering all five of the Fourier properties.

Generally speaking, Lewin's five Fourier properties test for balance. The properties FOURPROP(3-6) clearly demonstrate the notion of balance, in that they test whether or not a set is evenly distributed amongst the various instances of a given cycle (2-cycle, 3-cycle, etc.). FOURPROP(1-2) also test for balance, however it is less clear from Lewin's formulation how exactly they do so. Without some understanding of the Fourier transform, or to at least a graphical illustration, the notion of balance remains obscure. One of Ian Quinn's contributions, as we shall see, is precisely that: he provided a graphical representation of Lewin's Fourier properties,

showing how the properties judge whether or not a set is balanced amongst the various cyclic collections.

Though Lewin never discussed any connection between the z-relation and the five Fourier properties, the relationship can nevertheless be seen: that is, sets that are z-related always possess the same Fourier properties. The z-related pair [0146] and [0137], for instance, both satisfy none of the five properties. However, because the properties are formatted as yes/no binary tests that do not provide any description of by how much a set is unbalanced, Lewin's Fourier properties cannot uniquely describe the sets with the same interval vector (as does the Fourier transform proper). Instead, with Lewin's five properties, there are several cases in which sets with different interval vectors satisfy the same set of properties. Only with Quinn's reinterpretation of the five properties as Fourier balances, which also indicates imbalance as well as balance, does it become apparent that the Fourier transform and the interval vector are telling the same story in two different ways.

2.4.2 Quinn and the Fourier balances

In his 2004 dissertation, and in the subsequent republication as Quinn 2007/2008, Ian Quinn adopted the Fourier transform and used it to develop a "unified theory of chord quality in equal temperaments." His work stands in response to the years of work by other theorists on similarity measures—that is, measures that calculate in some fashion the 'distance' between two sets. The 'similarity measure' enterprise had been a long lasting one, starting with Allen Forte (1973) and

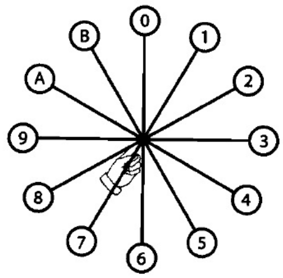
continuing throughout the 1990s.[19] In general, similarity measures had all been based on either the interval vector or subset inclusion, or both. Alternatively, Quinn sought to remedy some of the issues surrounding similarity measures by developing a theory of chord quality based on the Fourier transform. Rather than profiling sets by their interval vector or by their subsets, Quinn considered sets in terms of how they compare to the various cycles. The collections that are themselves cycles—such as [06], [048], etc.—are considered as primary prototypes; those that are composites of two or more cyclic collections are secondary and tertiary prototypes; and the other remaining non-cyclic collections are classified according to their imbalance in respect to the cycles, all of which is determined by the Fourier transform.

Quinn reinterpreted Lewin's five properties, redefining them as "Fourier balances." Unlike Lewin's properties, which simply indicate whether or not a set is balanced in respect to the cycles, Quinn's Fourier balances determine by how much a set is imbalanced. In other words, his method specifies to what degree a set does not have FOURPROP(x). The Fourier balances are to be understood as scales that are balanced on a fulcrum at the center, and that have pans that hold pitch classes, as shown in Example 2.13. The particular Fourier 'weight' of a pitch-class set can be found by dropping the pitch classes into the pans and seeing which way, and by how much, the scale tips over. Example 2.13a shows the Fourier balance that is associated with FOURPROP(1), and so forth.

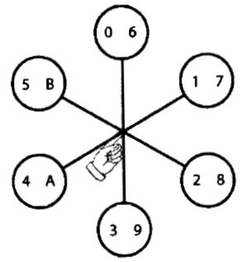
[19] Quinn's early article (2001) neatly summarizes the vast field of similarity measures, and details the various techniques that these measures employ, as well as the assumptions upon which they are based.

Example 2.13: The Fourier balances (Quinn 2004, 78-83).

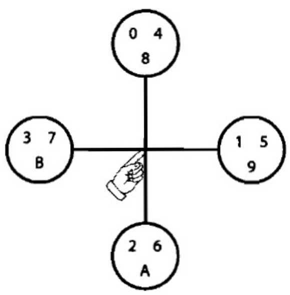
a) Fourier balance 1



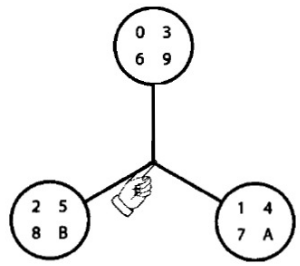
b) Fourier balance 2



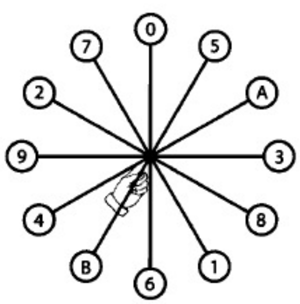
c) Fourier balance 3



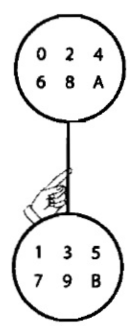
d) Fourier balance 4



e) Fourier balance 5



e) Fourier balance 6



The Fourier balances correspond to Lewin's Fourier properties, showing graphically the distribution of the pitch classes amongst the various cycles. Sets that

have FOURPROP(1), such as set classes [06], [048], [0167], etc. are balanced on Fourier balance 1 (Ex. 2.13a), and so on. Quinn also includes Fourier balance 5, even though Lewin had left out FOURPROP(5). Fourier balance 5 is structured similarly to Fourier balance 1, but it is instead based on the 5-cycle.

Unlike Lewin's Fourier properties, Quinn's balances also show imbalance. Pitch-class sets that do not have FOURPROP(*x*) will be unbalanced and tip in some direction. If the set is not balanced, the degree of imbalance (i.e. the angle at which it tips) can be calculated through a process of vector summation, where each pitch class is considered as a vector from the middle of the circle to the particular pitch class. For example, a vector pointing north represents pitch class 0, and a vector pointing south represents pitch class 6. Quinn explains vector summation in the following (2007, 41-42):

> [T]o add two arrows, move them relative to each other (taking care not to rotate them) so that the tail of one is attached to the head of another; then draw a new arrow from the free tail to the free head. The new arrow is the sum of the two.

Example 2.14 demonstrates a vector sum, showing the vector sum for the pc-set {0,1,4,6}. For each pitch class in the set, the arrows a(x) are joined end-to-end (in no particular order). The distance from the origin to the final point, marked with the heavy gray line, is the degree of imbalance (magnitude) measured in units of 'lewin' (abbreviated as Lw, which is pronounced as lew). In this case, the set [0146] has a tritone, which cancels out as a(0) and a(6) sum to zero, and thus [0146] is imbalanced

as much as is [03]. For both sets, the vectors sum to form a right triangle, and so the degree of imbalance is equivalent to the length of the hypotenuse, $\sqrt{2}$Lw ≅ 1.414Lw.

Example 2.14: Vector sum for the pc-set [0146], adapted from Quinn 2007.

The procedure of vector summation just described finds the degree of imbalance on Fourier balance 1. To find the degree of imbalance on the other Fourier balances, one can simply place the pitches into the pans of the desired Fourier balance, and derive the vectors according to the angles of the pans. The set {0,1,4,6} on Fourier balance 3, for example, would be imbalanced an amount equal to the magnitude of the sum of two vectors pointing north, one pointing south, and one pointing to the east, and thus would be equal to the magnitude of a single vector pointing north-east—that is, 1.414Lw. Alternatively, the degree of imbalance on Fourier balance *x* can be determined by just multiplying the set by *x* and considering it on Fourier balance 1—the effect is the same as placing the pitches in the different pans. For instance, the set {0,1,4,6} multiplied by 3, {0,0,3,6}, consists of the same set of vectors above, summing to a vector with a magnitude of 1.414.

To calculate the degree of imbalance by hand, the only data that must be pre-calculated are the point locations of the pitch classes on the circle in 2-dimensional Euclidean space, where each pitch class (pc) is located at the point:

FORMULA 2.1: $(\sin(2\pi * \text{pc}/m), \cos(2\pi * \text{pc}/m))$.

The point locations for the pitch classes of mod12 are shown in Table 2.2.[20]

Table 2.2: Point locations for mod12 in 2-dimensional Euclidean space.

pc	point location	pc	point location
0	$(0, 1)$	6	$(0, -1)$
1	$(1/2, \sqrt{3}/2)$	7	$(-1/2, -\sqrt{3}/2)$
2	$(\sqrt{3}/2, 1/2)$	8	$(-\sqrt{3}/2, -1/2)$
3	$(1, 0)$	9	$(-1, 0)$
4	$(\sqrt{3}/2, -1/2)$	10	$(-\sqrt{3}/2, 1/2)$
5	$(1/2, -\sqrt{3}/2)$	11	$(-1/2, \sqrt{3}/2)$

To derive the magnitude, first sum all the x-coordinates of the pitch classes together, and likewise with the y-coordinates. Then using these summed x- and y-coordinates, we apply the Pythagorean theorem to find the magnitude (in Lw), as in the following:

FORMULA 2.2:

$$X = \text{sum of x-coordinates}$$
$$Y = \text{sum of y-coordinates}$$
$$\text{Magnitude} = \sqrt{X^2 + Y^2}$$

[20] There is a major discrepancy between music theory and mathematics that is relevant here: the unit circle. In music theory, 0 degrees points north and the values increase moving clockwise, whereas in mathematics 0 degrees points east and the circle moves counterclockwise. Rather than attempting to introduce the standard mathematical formulas, I have decided to alter the usual mathematical formulas (swapping *x* and *y*) to accommodate the music-theoretical standards.

For example, the pitch-class set {0,1,4,6} has $X_{\{0,1,4,6\}} = 0 + 1/2 + \sqrt{3}/2 + 0 = (\sqrt{3}+1)/2$, and $Y_{\{0,1,4,6\}} = 1 + \sqrt{3}/2 - 1/2 - 1 = (\sqrt{3}-1)/2$, and thus the magnitude is:

$$\begin{aligned} &= \sqrt{(\tfrac{\sqrt{3}+1}{2})^2 + (\tfrac{\sqrt{3}-1}{2})^2} \\ &= \sqrt{\tfrac{3+2\sqrt{3}+1}{4} + \tfrac{3-2\sqrt{3}+1}{4}} \\ &= \sqrt{\tfrac{3+2\sqrt{3}+1+3-2\sqrt{3}+1}{4}} \\ &= \sqrt{\tfrac{8}{4}} \\ &= \sqrt{2} \end{aligned}$$

Again, we can see that the {0, 6} cyclic collection cancels out: $X_{\{0,6\}} = 0 + 0 = 0$ and $Y_{\{0,6\}} = 1 - 1 = 0$. Naturally, all cyclic collections cancel out in this manner, including transpositions of cycles, such as the cyclic collection {1, 5, 9}: $X_{\{1,5,9\}} = 1/2 + 1/2 - 1 = 0$ and $Y_{\{1,5,9\}} = \sqrt{3}/2 - \sqrt{3}/2 + 0 = 0$.

The same guidelines apply for calculating the degree of imbalance on different Fourier balances *x*. First the set is multiplied by *x*, and then the same summation process with the Pythagorean theorem finds the magnitude. For the set {0,1,4,6} on Fourier balance 4, for example, first multiply the set by 4, which makes {0,0,4,4}. Then calculate the sums of the x- and y-coordinates for {0,0,4,4}: $X_{\{0,0,4,4\}} = 0 + 0 + \sqrt{3}/2 + \sqrt{3}/2 = \sqrt{3}$, and $Y_{\{0,0,4,4\}} = 1 + 1 - 1/2 - 1/2 = 1$. Using the Pythagorean theorem, we then find that the magnitude is equal is the square root of 4, which, of course, is 2.

The magnitude is only part of the information given by the Fourier transform. In order to show some more points about the nature of the z-relation, let us now turn to review the Fourier transform as it is generally characterized in the sciences, introducing another aspect of the Fourier coefficient—the angle (or phase). By

incorporating the angle, it not only becomes possible to demonstrate more about the how the Fourier transform relates to the interval vector, but it also becomes possible to demonstrate an alternate explanation of z-related sets that involves supersets.

2.4.3 The Fourier coefficient

The degree of imbalance for a given Fourier balance x corresponds to what is referred to in the sciences as the 'magnitude (or amplitude) of the x-th Fourier coefficient.' The type of Fourier transform used with pitch-class sets is specifically the discrete Fourier transform (DFT), since pitch classes are regularly spaced points in a modular space. It could be argued that pitch classes also exist in between the cracks (so to speak), in what Clifton Callender (2007) calls a 'continuous space'. In such a space, where pitch classes such as π are possible, the pitch classes are not discrete, but instead fill the entire space from 0 to (but not including) m. Nevertheless, for the present purposes, I prefer to use the DFT in most cases, even if it involves transcribing the pitch classes to another modulus so that all the pitch classes are integers. The linear Fourier transform requires calculus, while the discrete version only requires basic algebra. However, if the pitch classes are irrational numbers, like π, then there is no choice: the linear Fourier transform must be used.

The discrete Fourier transform outputs an array of m (modulus) Fourier coefficients, where each coefficient is written as $\mathbb{F}_x$ where x is an integer and $0 \leq x <$ m. Essentially, a coefficient is exactly what was explained before: it is a vector from the origin of an axis to the end point of the summed pitch-class vectors, as described by Quinn. Each coefficient has a magnitude (length, or amplitude), and an angle in

radians (where 2π radians completes the circle). Usually, the coefficient is expressed as a complex number ($a + ib$, where i is the square root of -1), though it is also often written in polar form as an ordered duple (A, φ) indicating the magnitude (amplitude) and angle (phase). However it may be notated, it should be remembered that the coefficient is essentially a vector, which has a magnitude and an angle. For this study, I will use an slightly different notation, where $|\mathbb{F}_x|$ (the absolute value of the coefficient) indicates the magnitude, and $\arg(\mathbb{F}_x)$ the angle (arg being a specific formula from mathematics). Though this notation may be somewhat messier, I prefer it since it clearly shows that the Fourier coefficient is technically a complex number.

The magnitude $|\mathbb{F}_x|$ is equivalent to the degree of imbalance on the Fourier balance x, and calculated with the method given above—that is, 1) multiply the set by x; 2) find the x- and y-coordinates of the pitch classes in the multiplied set; and 3) sum all the x-coordinates and y-coordinates, and use the Pythagorean theorem to find the magnitude of the coefficient. The magnitude of the 0-th Fourier coefficient ($|\mathbb{F}_0|$) is always equivalent to the cardinality of the pitch-class set, and the x-th Fourier coefficient (where $x \neq 0$) has the same magnitude (though possibly different angle) as the $(m-x)$-th coefficient.

To calculate the angle of the coefficient $\arg(\mathbb{F}_x)$, on the other hand, we need a new formula (and a calculator). Using the sums of the x- and y-coordinates of all the pitch classes in the set (X and Y), the angle (in radians) is found with:[21]

---

[21] To account for the discrepancy between the music theoretical and the mathematical definitions of the unit circle, the given atan2 formula has the X and Y switched in respect to the standard definition.

FORMULA 2.3: $\arg(\mathbb{F}_x) = \text{atan2}(X, Y)$

$$= 2\arctan\frac{X}{\sqrt{X^2+Y^2}+Y}$$

With some of the simpler cases, however, we can intuit what the angle will be. For example, with the $\mathbb{F}_1$ of set {0, 1, 4, 6} in mod12, the {0} and {6} cancel out, leaving behind the dyad {1, 4}. Since the coefficient must lie half way in between the pitch classes of the [03] dyad (using the shorter of the two paths), the angle of the coefficient for the {1, 4} dyad is easily determined: the coefficient crosses through the pitch-class 2.5, and thus the angle is 2π * 2.5/12 = 5π/12.

As an example of a complete DFT, let us consider the DFT of the set {0, 1, 4, 6} in mod12. The DFT outputs an array with 12 coefficients. Though technically the coefficients are complex numbers, for simplicity we can write the DFT array as a pair of arrays, one with the magnitudes and the other with the angles. For the set {0, 1, 4, 6} the arrays are as follows:

$A = \{0, 1, 4, 6\}$, and $A \subseteq \mathbb{Z}_{12}$
$|\mathbb{F}_x(A)| = [4, \sqrt{2}, 2, \sqrt{2}, 2, \sqrt{2}, 2, \sqrt{2}, 2, \sqrt{2}, 2, \sqrt{2}]$
$\arg(\mathbb{F}_x(A)) = [0, 5\pi/12, 0, \pi/4, \pi/3, 13\pi/12, 0, 11\pi/12, 4\pi/3, 3\pi/4, 0, 7\pi/12]$

Each of the *m* coefficients can be referenced either just by the magnitude or angle, such as $|\mathbb{F}_0(A)| = 4$ or $\arg(\mathbb{F}_3(A)) = 1/4\pi$, or in polar notation, giving both the magnitude and angle of the coefficient, such as $\mathbb{F}_0(A) = (0, 0)$ and $\mathbb{F}_3(A) = (1.414, 1/4\pi)$. In cases that consider only the magnitudes of the coefficients, I will prefer the former.

Now that we have established an expanded definition of the Fourier transform that shows both the magnitude and angles of the Fourier coefficients, let us now turn to look how the Fourier transform and the interval vector show the same phenomenon in two different ways, and why z-related sets tend to partition into identical subsets.

2.5 The Fourier transform, interval vectors and z-related sets

Though Quinn was not specifically concerned with z-related sets, his Fourier balances expose for the first time in music theory a crucial feature of the z-relation: sets with the same interval vector have the same degree of imbalance on all five of the Fourier balances. That is, two sets with the same magnitudes on all Fourier balances are either of the same set class or z-related. Using Soderberg's notation presented in Chapter 1, we can now formally express the relationship between the Fourier balances and the interval vector as the following:

THEOREM 2.1: If V(A) = V(B), then
$|\mathbb{F}_x(A)| = |\mathbb{F}_x(B)|$ for all x, where $0 \leq x < m$.

THEOREM 2.2: If $|\mathbb{F}_x(A)| = |\mathbb{F}_x(B)|$ for all x, where $0 \leq x < m$, then
V(A) = V(B).

Since the Fourier transform describes the same phenomenon as does the interval vector, we can also rewrite some of Soderberg's 'basic vector theorems', such as Soderberg's theorems 2.2 and 2.6 (Soderberg 1995, 81-82), substituting the Fourier transform for the interval-class vector:

THEOREM 2.3 (from Soderberg's theorem 2.2):

$|\mathbb{F}_x(A)| = |\mathbb{F}_x(T_yA)| = |\mathbb{F}_x(T_zIA)|$ for all x, where $0 \leq x < m$

THEOREM 2.4 (from Soderberg's theorem 2.6):

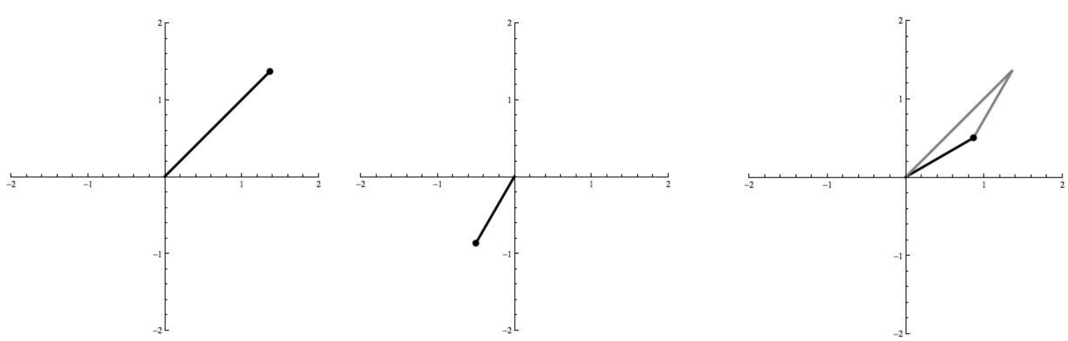
$\mathbb{F}_x(A \uplus B) = \mathbb{F}_x(A) + \mathbb{F}_x(B)$ for all x, where $0 \leq x < m$

Theorem 2.3 states that the Fourier transform of a set will be equivalent to that of all transpositions and inversions of the set. Theorem 2.4 addresses the Fourier transform of unions of sets (sums of multisets), showing associativity (in the mathematical sense). Since the Fourier transform reflects the same information as the interval vector, the formulas hold true with either interval vectors or Fourier coefficients.

Theorem 2.4 becomes apparent when visualized on a Euclidean space. To show this, Example 2.15 graphs the 1st Fourier coefficients of the sets A = {0,1,4}, B = {7} and the union A ∪ B = {0, 1, 4, 7}. As indicated by the grays lines in 2.15c, the coefficient of the sum of A and B is equal to the sum of the two vectors in Exs. 2.15a and b.

Example 2.15: Fourier coefficient ($\mathbb{F}_1$) of the union of two sets, A = {0,1,4} and B = {7} in mod12.

a) $\mathbb{F}_1(A)$ b) $\mathbb{F}_1(B)$ c) $\mathbb{F}_1(A \cup B)$



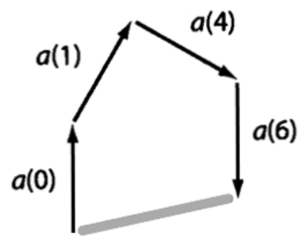
Given Theorem 2.4, and that z-related sets have equal magnitudes on the Fourier balances (i.e., $|\mathbb{F}_x A| = |\mathbb{F}_x B|$ for all x, where $0 \leq x < m$), it follows that if two z-related sets share cyclic collections as subsets (such as [06] or [048] in mod12), then the two subsets that remain once the cyclic collections are extracted from each z-related set (the residual 'remainder' sets) will have the same magnitude on Fourier balance 1. Cyclic collections are balanced on Fourier balance 1, having 0Lw; thus, when the cyclic-collection subsets are extracted, the magnitude of each of the two z-related sets on Fourier balance 1 is left unchanged, and the magnitude of the remainder set is equivalent to the magnitude of the z-related set. In most cases (specifically, with the Type 1 pseudohomometric sets), the two remainder sets (of the two z-related sets) are of the same set class.

As an example of a z-related pair that shares a partition based on cyclic collections (a partition that I call the *Fourier partition*), take the z-related pair {0, 1, 4, 6} and {0, 1, 3, 7}. Example 2.16 gives the vectors sums of each set. Both sets have a tritone that cancels out on Fourier balance 1, and thus the two remainder sets (being in both cases the set class [03]) have a magnitude of 1.414Lw. Example 2.17 graphs the Fourier coefficients of the two sets, as well as for the set {0, 3}. The magnitudes of the three coefficients are equivalent since the cyclic collections in the two all-interval tetrachords in effect cancel out, leaving behind a remainder set of set class [03]. However, since transposition affects the angle of the coefficients, the angles of the coefficients of the two all-interval tetrachords (in Examples 2.17b and c)

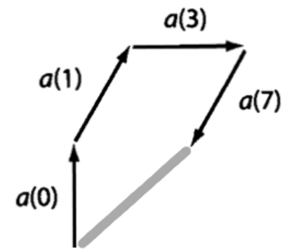
are not necessarily the same, depending on which two all-interval tetrachords we compare (see below for more on how transposition affects the Fourier coefficient).

Example 2.16: Vector sums for sets {0,1,4,6} and {0,1,3,7} using Quinn's diagrams.

{0,1,4,6}

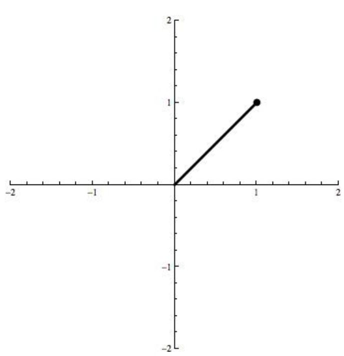


{0,1,3,7}

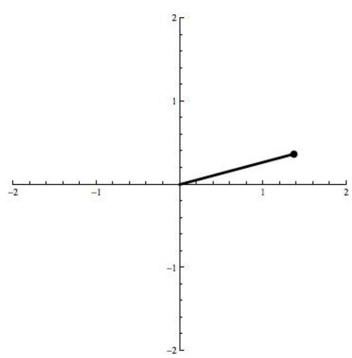


Example 2.17: Fourier coefficients ($\mathbb{F}_1$) of {0, 3}, {0,1,4,6} and {0,1,3,7}.

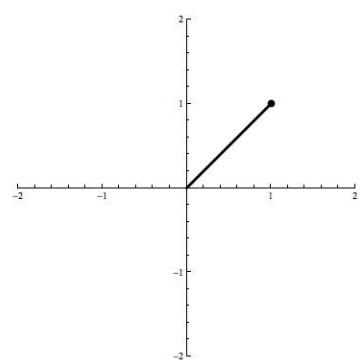
a) $\mathbb{F}_1(\{0, 3\})$



b) $\mathbb{F}_1(\{0, 1, 4, 6\})$



c) $\mathbb{F}_1(\{0, 1, 3, 7\})$



Extracting cyclic collections from a pair of z-related sets leaves two remainder sets that have the same degree of imbalance on Fourier balance 1. Most cases are Type 1 pseudohomometric sets, with identical remainder sets. Some others are Type 2 pseudohomometric sets, with identical cyclic collections and homometric remainder sets. The Type 3 pseudohomometric sets have identical cyclic-collection subsets, but

by definition they have remainder sets that are neither identical nor homometric. These are generally found among the homometric triples, quadruples, etc. (rather than pairs), such as the sets $U_{1\text{-}3}$ in mod16 (shown in Example 2.10 above):

U_1 = {0, 1, 4, 5, 7, 8, 10} => [0, 8] ∪ [0, 3, 4, 6, 9]
U_2 = {0, 1, 2, 4, 5, 8, 11} => [0, 8] ∪ [0, 1, 3, 4, 10]
U_3 = {0, 3, 4, 5, 6, 9, 12} => [0, 8] ∪ [0, 3, 4, 6, 9]

The two subsets, set classes [0, 3, 4, 6, 9] and [0, 1, 3, 4, 10], have $\mathbb{F}_1$ coefficients of the same magnitude, but they are not homometric. However, though set U_2 does not share with the other two sets a partition based on the common cyclic collections, it does share a partition with the other two sets that does not involve cyclic collections (see Example 2.18).Nevertheless, though the subsets [0, 3, 7] and [0, 3, 5, 10] or [0, 1, 2, 5] are shared amongst U_2 and its z-partners, the subsets cannot be used to form a valid algebraic formula.

Example 2.18: Alternate partition of sets in homometric triple in mod16.

U_1 = {0, 1, 4, 5, 7, 8, 10} => [0, 3, 7] ∪ [0, 3, 5, 10]
U_2 = {0, 1, 2, 4, 5, 8, 11} => [0, 3, 7] ∪ [0, 3, 5, 10]

U_2 = {0, 1, 2, 4, 5, 8, 11} => [0, 3, 7] ∪ [0, 1, 2, 5]
U_3 = {0, 3, 4, 5, 6, 9, 12} => [0, 3, 7] ∪ [0, 1, 2, 5]

Since cyclic collections are balanced, extracting them from a set has no effect on the degree of imbalance. Accordingly, it naturally follows that if there is a shared partition between two z-related sets (where all subsets are identical), then it is likely to involve cyclic collections. To say that a pair of z-related sets shares a partition involving cyclic-collections, however, does not presuppose that z-related sets share

all cyclic collections. There are a few hexachordal pairs in mod12, for instance, where one set has a [048] subset while the other does not. In fact, there is only one hexachordal z-related pair where both sets have a [048] subset ([013478] and [012569]), but there are 5 hexachordal z-related pairs where only one set as a [048] subset. However, the Fourier transform does not tell us that the two sets of a z-related pair share *all* cyclic-collection subsets. Rather, it tells us that when the greatest number of cyclic collections (involving the most pitch classes) are extracted from the two z-related sets, the subsets left behind will have the same degree of imbalance on Fourier balance 1. Consequently, if one z-related set partitions into one cyclic collection plus a set, so too will its z-related partner.

To demonstrate further the notion of cyclic-collection content, take for example the z-related pair [012478] and [012568]—the all-trichord hexachord (ATH) and its complement. Though the ATH has a [048] subset, the complement does not. However, the [048] + [026] partition is not the partition of the ATH that involves the most pitch classes in cyclic collections (only 3 pitch classes are involved in the [048]). Rather, the partition [06]+[06]+[04] is the partition with the most pitch classes in cyclic collections (total of 4 pitch classes), and is thus the partition that is shared between the ATH and its complement.

2.5.1 Effects of transposition/inversion on Fourier coefficients

Though the magnitudes of the Fourier coefficients are always the same between transpositions and inversions of a set (theorem 2.1), the angles may be different. Transposition of a set affects the coefficient by rotating it around the origin

by x times the level of transposition y. Since for each of the m coefficients of the DFT the set is multiplied by x before the set is 'weighed' on the Fourier balance, each increment of the transposition y rotates the coefficient by x. If a set (in mod12) is transposed at T_3, for instance, the coefficient $\mathbb{F}_5(T_3(A))$ is rotated by $xy = 5*3 = 3$ steps in relation to $\mathbb{F}_x(A)$. Formally, we write this as:

FORMULA 2.4: $\arg(\mathbb{F}_x(T_yA)) = \arg(\mathbb{F}_x(A)) + 2\pi*(yx/m) \quad \mathrm{mod}2\pi$

For example, compare Examples 2.19a and 2.19b, which are the $\mathbb{F}_1$ coefficients of a set and T_3 of the set. As can be seen, the vector in 2.19b is the vector in 2.19a rotated by $2\pi*(yx/m) = 2\pi*(1*3/12) = 2\pi*(1/4)$ radians, i.e. a quarter circle.

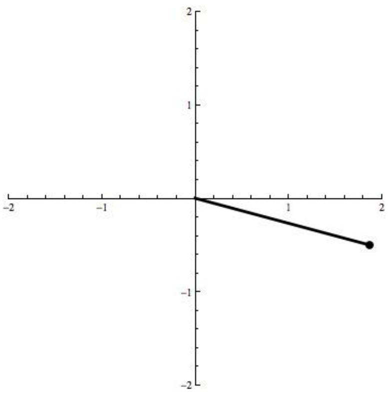
Inversion, on the other hand, flips the coefficient about the y-axis (compare Examples 2.19a and 2.19c). Here inversion means I_0, which is equivalent to multiplying each pitch class in the set by –1. Accordingly, the relationship between the Fourier coefficients of sets related by inversion is expressed as:

FORMULA 2.5: $\arg(\mathbb{F}_x(IA)) = -\arg(\mathbb{F}_x(A)) = 2\pi - \arg(\mathbb{F}_x(A))$

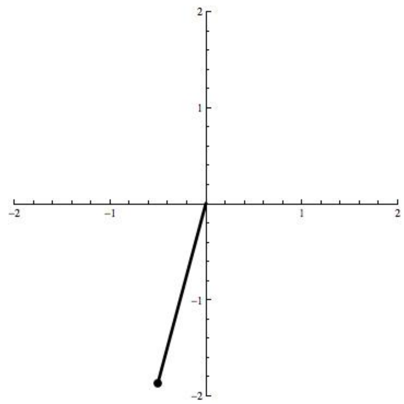
Inversion at another index besides 0, such as I_8, is to be understood as inversion followed by transposition, as in T_8I. Inversion/transposition thus first flips the Fourier coefficient around the y-axis, and then rotates it according to Formula 2.4. Thus, Example 2.19d, which is the graph of $\mathbb{F}_1(T_3IA)$, results from flipping the vector of Example 2.19a around the y-axis (becoming Example 2.19c), and then rotating the vector clockwise by a quarter circle, i.e. $2\pi*xy/m = 2\pi*(1 * 3/12) = 2\pi*(1/4)$ radians.

Example 2.19: Effects of transposition and inversion on the Fourier transform: some graphs of the coefficient $\mathbb{F}_1$ of transpositions and inversions of a set.

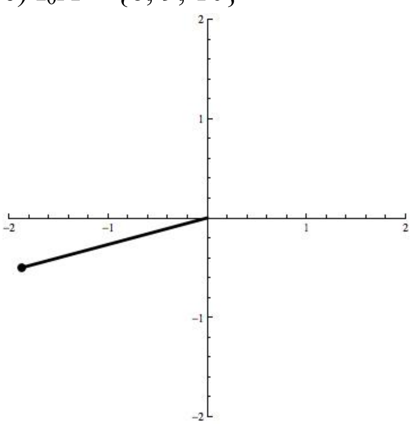
a) A = {2, 3, 6}



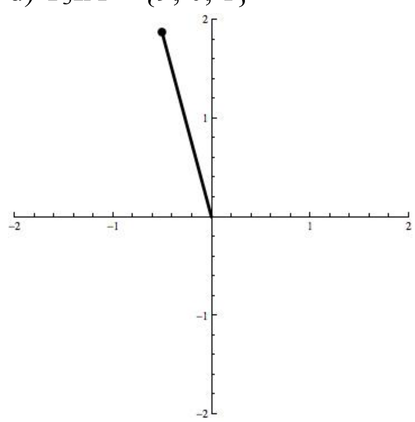
b) T_3A = {5, 6, 9}



c) I_0A = {6, 9, 10}



d) T_3IA = {9, 0, 1}



2.5.2 Sets with opposite-pointing Fourier coefficients

At this point, we can now begin working towards a proof of the proposition that for any pair of z-related sets there will always be some transposition and/or inversion of the two sets such that the union of the sets forms a superset that has FOURPROP(x), and thus is balanced on Fourier balance x—a proposition that explains Morris and Soderberg's observation (see Chapter 1) that the two all-interval

tetrachords can join to make an octatonic collection. But before we consider z-related sets, let us first consider any set that does not have FOURPROP(*x*). Given that the set has a coefficient $\mathbb{F}_x$ that points in some direction, there are some transposition(s) and inversion(s) of the set that have a coefficient $\mathbb{F}_x$ that points in the opposite direction. The number of possible transpositions that have such coefficients is dependent on *x*. Specifically, for any set there are G = GCD(x, m) transpositions that have coefficients that point in the opposite direction from the coefficient of the original set.[22] Given that k is some integer, the various levels of transposition (y) that generate such a coefficient are found with the following:

FORMULA 2.6:

$$
\begin{aligned}
y &= \frac{m}{2G} + \frac{\mathrm{k}m}{G} && \mathrm{mod}\, m \\
&= \frac{m}{2G} * (2\mathrm{k} + 1) && \mathrm{mod}\, m
\end{aligned}
$$

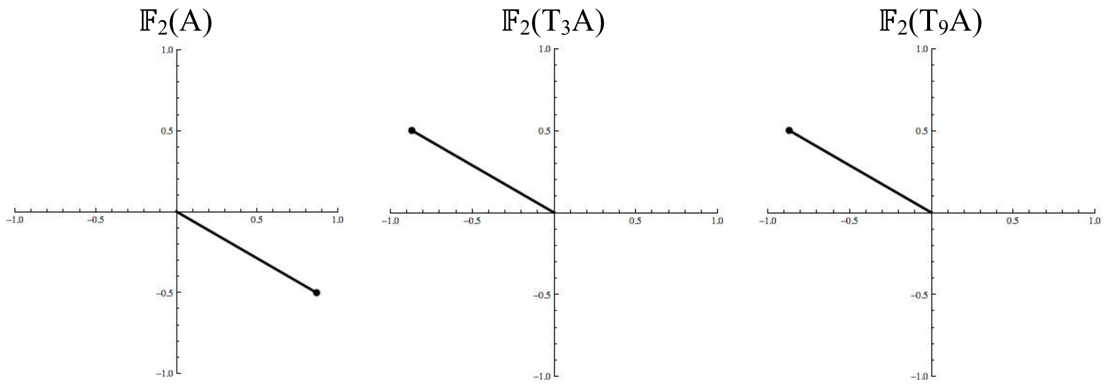
For example, for $\mathbb{F}_2$ in mod12, G = GCD(2, 12) = 2, and thus the levels of transposition that yield a set with a coefficient pointing in the opposite direction would be T_3 and T_9. Example 2.20 illustrates this notion, showing $\mathbb{F}_2$ of the set {2, 3, 6}, and $\mathbb{F}_2$ of the transpositions of the set at T_3 and T_9.

For any given set A there will also be G transpositions of the inversion of the set that have coefficients that point in the opposite direction. The same formula above applies for the transposition portion of the T_yI transformation, but now we must also account for the effect of the inversion on the Fourier coefficient, which unfortunately involves more calculation since we must know the value of arg($\mathbb{F}_x$(A)).

---

[22] GCD indicates greatest common divisor.

Example 2.20: Transpositions of the set A = {2, 3, 6} that have $\mathbb{F}_2$ coefficients pointing in opposite direction from $\mathbb{F}_2$ coefficient of set A.



Nevertheless, given $\arg(\mathbb{F}_x(A))$, the levels of transpositions of the inversion (T_yI) that have opposite-pointing coefficients are as follows:

FORMULA 2.7:

$$\begin{aligned} y &= \tfrac{m}{2G} + \tfrac{\mathrm{k}m}{G} + 2\arg(\mathbb{F}_x(\mathrm{A})) * \tfrac{m}{x2\pi} \qquad \bmod m \\ &= \tfrac{m}{2G} * (2\mathrm{k}+1) + \arg(\mathbb{F}_x(\mathrm{A})) * \tfrac{m}{x\pi} \quad \bmod m \end{aligned}$$

For example, let us reconsider the set given above, A = {2, 3, 6}. The angle of the coefficient, $\arg(\mathbb{F}_2(A))$, is $2\pi/3$, and thus according to the formula the levels of T_yI that yield sets with opposite-pointing $\mathbb{F}_2$ coefficients are T_1I and T_7I, which yield the sets {7, 10, 11} and {1, 4, 5}.

If T_y or T_yI generates a set that has a coefficient pointing in the opposite direction from the coefficient of the original set, then the union (multiset sum) of the set and T_y (or T_yI) of the set (using theorem 2.4) will necessarily be a superset that is balanced and that has FOURPROP(x), since the coefficients will cancel out each other and sum to a null vector with zero magnitude. For example, take the first two

sets in Example 2.20, sets {2, 3, 6} and {5, 6, 9}. The sum of these two multisets, the multiset {2, 3, 5, 6, 6, 9}, has a $\mathbb{F}_2$ coefficient of zero magnitude, and therefore it has FOURPROP(2)—that is, the multiset has the same number of common pitches with any 0167-set as it does with T_3 of the 0167-set. Let us express the proposition formally as:

THEOREM 2.5: if $y \equiv m/(2G) \bmod m/G$, then
$\mathbb{F}_x(A) + \mathbb{F}_x(T_yA) = 0$, and thus
$(A \uplus T_yA)$ has FOURPROP(x).

THEOREM 2.6: if $y \equiv m/(2G) + 2\arg(\mathbb{F}_x(A))*m/(x2\pi) \bmod m/G$, then
$\mathbb{F}_x(A) + \mathbb{F}_x(T_yIA) = 0$, and thus
$(A \uplus T_yIA)$ has FOURPROP(x).

Now let us consider z-related sets. Since the sets of a z-related pair always have Fourier coefficients of the same magnitudes, we can apply the same methodology as above; that is, for any $\mathbb{F}_x$ there will be some transposition and/or inversion of the two sets, such that the sum of the two z-related multisets has a coefficient $\mathbb{F}_x$ with zero magnitude. Assuming that one of the two sets (A) is held constant while the other set (B) is transposed or inverted, there will be *G* transpositions of (and *G* transpositions of the inversion of) set B, such that set B will have a coefficient $\mathbb{F}_x$ that points in the opposite direction of the coefficient of set A. To find these levels of transposition, it is necessary to calculate the coefficients for the two sets, and to find the difference in angle between the two coefficients. The difference in angle between the two coefficients (φ) is:

FORMULA 2.8: $\varphi = \arg(\mathbb{F}_x(A)) - \arg(\mathbb{F}_x(B))$

The level of transposition that takes B to the position where it has a coefficient pointing in the same direction as that of A is dependent on x. Again, this is because the complete circle, all 2π radians, fills a different amount of transpositional space depending on the cycle we are considering. For each coefficient $\mathbb{F}_x$, the unit circle (2π) fills the space of one unit of an m/x cycle, and thus T_{12} of the set rotates the $\mathbb{F}_1$ coefficient 2π radians, T_6 of the set rotates the $\mathbb{F}_2$ coefficient 2π radians, and so on. Accordingly, we translate the angle φ between the two coefficients to the transposition level u:

FORMULA 2.9: $u = (\arg(\mathbb{F}_x(A)) - \arg(\mathbb{F}_x(B))) * m/(x2\pi)$

The levels of transposition of set B that yield coefficients that point in the opposite direction are thus $u+y$ for all y that satisfy formulas 2.6 and 2.7.

Let us take as an example two all-interval tetrachords, pitch-class sets A = {0, 1, 4, 6} and B = {0, 2, 3, 8}. The 1st Fourier coefficients of the two sets are $\mathbb{F}_1(A) = (\sqrt{2}, 5\pi/12)$ and $\mathbb{F}_1(B) = (\sqrt{2}, \pi/4)$. The difference in angle between the two coefficients is $\mathbb{F}_1(A) - \mathbb{F}_1(B) = 5\pi/12 - \pi/4 = \pi/6$. In mod12, the difference translates to a transposition of $u = \pi/6 * 12/(2\pi) = 1$, and thus if we transpose B by T_1, the coefficient $\mathbb{F}_1(T_1B)$ will be pointing in the same direction as $\mathbb{F}_1(A)$. To find the levels of transposition that rotates the coefficient of B so that it points in the opposite direction, we then solve for y. In this case, since $x = 1$, it follows that $G = 1$, and the sole solution for y is 6. Finally, $u+y = 7$, and thus T_7B (the set {3, 7, 9, 10}) has a 1st Fourier coefficient that points in the opposite direction than that of set A. Therefore,

the multiset sum A $\uplus$ T_7B forms a pitch-class set that has FOURPROP(1)—a set that is the union of tritones and/or augmented triads—namely, the octatonic collection {0, 1, 3, 4, 6, 7, 9, 10}.

The property described here—that z-related sets can be arranged with opposite-pointing coefficients so that the sum of the sets is 'balanced'—holds true for any pair of z-related sets in any modulus, as well as, of course, sets of the same set class. Naturally, any z-related set can be transposed or inverted so that some particular coefficient is pointing in the opposite direction from that of its z-related partner. This remains true even with the more unusual cases, such as the tetrachordal z-related pair in mod13, set classes [0146] and [0237]. However, in such cases the results may be less than useful for musical purposes, since the transpositions/inversions may involve real-number intervals rather than integer intervals. For this reason, I will leave such cases for the interested reader.

2.6 Conclusion

Callender and Hall's recognition of the connection between z-related sets and homometric sets has opened a door of new explanations of the z-relation in music theory. The Fourier transform, which crystallographers use to derive sets, shows that two z-related sets share the same cyclic-collection content. Since two z-related sets have equal-valued magnitudes on all Fourier balances, when any cyclic collection is extracted, the two sets left behind will invariably have the same degree of imbalance on Fourier balance 1. This means that if one z-related set is partitioned into the maximal number of cyclic collections plus a set, the z-related partner will also

partition into the same cyclic collections plus a set. Furthermore, from crystallography we learn that some z-related sets (including the mod12 z-related sets) are expressible as short algebraic expressions based on subset unions of common subsets. Though not all homometric sets can be explained with such formulas, the majority of them can.

The Fourier transform also shows that for each pair of z-related sets, there is some transposition of the sets so that the two z-related sets sum to a superset that is itself a union of cyclic collections, a property that Soderberg exploited in his Q-grids—the Q-grid is simply a large pitch-class collection comprised entirely of cyclic collections. Any z-related pair (even the homomorphs) can be made into a set with FOURPROP(1) under certain transpositions.

Though the Fourier transform offers a wealth of information beyond that provided by the interval vector, it does not, however, answer all concerning z-relation formulas. As to the question of which sets can be adjoined to cyclic collections to make z-related sets in algebraic formulas—that is, the sets that can act as valid remainder sets—it is necessary to take a different approach.

3. An Algebraic Approach to Z-Related Sets

As discussed in the previous chapter, crystallographers have shown that certain homometric sets can be expressed as algebraic equations. The purpose of this chapter is to propose more equations based on the model given by crystallographers, and to reformulate these equations so that they express relationships that are relevant to music analysis. As we shall see, all of the z-related pairs in mod12 can be expressed with algebraic equations, and some with multiple equations. Because of their generality, the algebraic equations reveal common transformational relationships shared by numerous pairs of z-related sets. Some of the developed algebraic equations can be further rearranged into other forms (by transposing or inverting either side of the equation), showing alternative ways in which z-related sets relate. In addition, we shall also see that there are inter-cardinality relationships between certain z-related sets, since some z-related pairs can be 'pumped'—a process that adds pitch classes to a set—to make larger cardinality z-related pairs.

As shown in Chapter 2, the z-related sets in mod12 are all Type 1 pseudohomometric sets: they have cyclic collections, and they partition into identical cyclic-collection and remainder (residual) subsets. These cycle-based partitions—what I call *Fourier partitions*—correspond to Quinn's Fourier balance 1 (and Lewin's FOURPROP(1)), in that the cyclic collections ([06] and [048]) are balanced (0Lw) and cancel out. Since z-related sets are equally imbalanced on the Fourier balances (having the same magnitudes on all balances), extracting cyclic-collection subsets from a pair of z-related sets leaves behind 'remainder' subsets with equivalent

magnitudes, which in most cases means that the remainder subsets are of the same set class. If the sets of a z-related pair have remainder subsets of the same set class, then the pair is Type 1 pseudohomometric, and thus it is possible to express the z-relation algebraically.

Of the 23 pairs of z-related sets in mod12, 21 pairs are based on the [06] cycle, and the remaining 2 are based on the [048] cycle. That is, if one takes a pair of z-related sets and extracts the maximal number of [06] or [048] cyclic collections out of each set in the pair, the same remaining subset will be left behind. If one extracts the [06] out of each of the all-interval tetrachords, for example, in both cases the remaining subset will be [03]. Because z-related sets partition in this way, we can identify a z-related pair by its Fourier partition, as the cyclic collections ([06] or [048]) and the non-cyclic remainder collection in each set. Table 3.1 shows all 23 pairs of z-related sets in mod12 and the Fourier partitions for each pair.[1]

Some of the z-related pairs shown in Table 3.1 partition into only one cyclic collection plus a set, while others partition into multiple cyclic collections plus a set. The tetrachord/octachord and pentachord/heptachord complemental pairs all partition into the subsets of the same set class, except that the larger pair has additional [06] collections positioned in such a way so that the z-relation is maintained—that is to say, [06] cycles cannot necessarily be added ad hoc to make complementary sets without at least fulfilling certain criteria (as described by pumping).

[1] The ordering of the sets in the table is based on the normal form defined by Brinkman (1986), in which sets are considered by their binary representations. The prime form is the transposition/inversion of the set that is the most compressed judging from the right.

Table 3.1: The 23 pairs of z-related sets in mod12, expressed as the union of one or more cyclic collections plus some remainder set.

Tetrachords

Z-related pair		**Fourier partition**
[0146] 4-Z15	[0137] 4-Z29	[06] + [03]

Pentachords

Z-related pair		**Fourier partition**
[01356] 5-Z12	[01247] 5-Z36	[06] + [024]
[01457] 5-Z18	[01258] 5-Z38	[06] + [015]
[01348] 5-Z17	[03458] 5-Z37	[048] + [02]

Hexachords

Z-related pair		**Fourier partition**
[012356] 6-Z3	[012347] 6-Z36	[06] + [0124]
[012456] 6-Z4	[012348] 6-Z37	[06] + [0134]
[012457] 6-Z11	[012358] 6-Z40	[06] + [0135]
[013457] 6-Z10	[023458] 6-Z39	[06] + [0125]
[012467] 6-Z12	[012368] 6-Z41	[06] + [06] + [02]
[013467] 6-Z13	[012369] 6-Z42	[06] + [06] + [01]
[012567] 6-Z6	[012378] 6-Z38	[06] + [06] + [03]
[013468] 6-Z24	[012469] 6-Z46	[06] + [0237]
[012568] 6-Z43	[012478] 6-Z17	[06] + [06] + [04]
[013568] 6-Z25	[012479] 6-Z47	[06] + [0247]
[023568] 6-Z23	[023469] 6-Z45	[06] + [06] + [02]
[013478] 6-Z19	[012569] 6-Z44	[06] + [0148]
[013578] 6-Z26	[012579] 6-Z48	[06] + [0358]
[013569] 6-Z28	[013479] 6-Z49	[06] + [06] + [04]
[023679] 6-Z50	[014679] 6-Z29	[06] + [06] + [05]

Heptachords

Z-related pair		**Fourier partition**
[0123568] 7-Z36	[0123479] 7-Z12	[06] + [06] + [024]
[0124578] 7-Z38	[0145679] 7-Z18	[06] + [06] + [015]
[0134578] 7-Z37	[0124569] 7-Z17	[048] + [06] + [02]

Octachords

Z-related pair		**Fourier partition**
[01235679] 7-Z29	[01234689] 7-Z15	[06] + [06] + [06] + [03]

The representative algebraic formulas to be developed will vary depending on how many cyclic collections are in the partition. The formulas introduced by crystallographers (particularly Patterson 1944 and Rosenblatt 1984, as discussed in Chapter 2), and later presented by Clifton Callender and Rachel Hall (2008), account for some (but not all) of the z-related pairs that have only one cyclic collection in the partition. By changing the criteria for the pitch-class set that is paired with the single cyclic collection (i.e., the remainder set), I will show how these formulas can be generalized to a single formula that accounts for a large majority of single-cycle z-related pairs. To account for the other pairs that partition into more than one cyclic collection, which cannot be described by the previous formulas, I employ two methods: first, a method that generates z-related pairs from smaller pairs, what O'Rourke, et al. (2008) call "pumping"; and second, an original formula, which describes what I call 'reciprocal set union', to account for certain pairs that possess two cyclic collections and that together form an aggregate. First, let us begin by outlining the algebraic expressions (and the corresponding z-transformations) for the z-related sets based on only one cyclic collection.

3.1 Z-related sets with one cyclic-collection subset

Of all of the z-related sets, the ones that are the simplest to express algebraically are those that have a single cyclic-collection subset. Most of the examples of homometric sets seen thus far have been sets of this kind, partitioning into only one cyclic collection plus a remainder set. To develop a general formula for the single-cycle z-related sets, let us start by considering the formula given by Clifton

Callender and Rachel Hall in their 2008 SMT paper. The formula (which is their case 1 of four cases of the z-relation) is a general formula for the all-interval tetrachords, and derives from Patterson 1944 and Bullough 1961. As Callender and Hall point out, the following expression holds true for any value of x, where $x > 0$ mod3:

$$\{0, x, x + 3, 6\} \;\mathcal{Z}\; \{0, x + 3, 6, x + 6\} \text{ mod12}$$

The formula captures the essence of all z-relation formulas based on one cyclic collection, in that there is a cyclic collection ($\{0, 6\}$) shared between the two sides of the formula and another subset based on a variable x—$\{x, x + 3\}$ or $\{x + 3, x + 6\}$. However, in order to change the formula into a form that better exposes the Fourier partition (the partition that extracts cyclic-collection subsets), let us perform a couple transformations—a transposition and an inversion. First, if we transpose the right hand side by 6, we get:

$$\{0, x, x + 3, 6\} \;\mathcal{Z}\; 6 + \{0, x + 3, 6, x + 6\}$$
$$\{0, x, x + 3, 6\} \;\mathcal{Z}\; \{6, x + 9, 0, x\}$$

which can be rewritten as:

$$\{0, x, x + 3, 6\} \;\mathcal{Z}\; \{0, x - 3, 6, x\}$$

Next, let us multiply the right-hand side by –1, which inverts the set around 0.

$$\{0, x, x + 3, 6\} \;\mathcal{Z}\; \{0, -x + 3, 6, -x\}$$

Now, let us rewrite the formula using subset unions, dividing each side into two subsets—one that has the elements based on the variable x, and the other with the static elements, 0 and 6.

$$\{0, 6\} \cup \{x, x + 3\} \;\mathcal{Z}\; \{0, 6\} \cup \{-x, -x + 3\}$$

Finally, let us extract the variable x out of each of the latter subsets in the two subset-unions:

FORMULA 3.1: $\{0, 6\} \cup x + \{0, 3\} \;\; \mathcal{Z} \;\; \{0, 6\} \cup -x + \{0, 3\} \qquad x > 0 \text{ mod3}$

In this final form, the formula clearly exhibits the same structure as that yielded by the Fourier transform: the two z-related sets both partition into the same cyclic collection and remainder set. The formula also shows that while the {0, 6} remains static between the two sides, the {0, 3} is transposed by x on one side and by $-x$ on the other. In this regard, the two sides of the formula share a positive/negative-transposition relationship, albeit a partial one since only certain subsets are affected by the transposition. With this particular formula, if x is some integer (not divisible by 3, or else the two sides will be of the same set class), then one side will be of the set class [0146], and the other of set class [0137].

Formula 3.1 (the interpolation of Callender and Hall's formula that reflects the Fourier partition) is the starting point for a general formula that accounts for a large number of single-cycle z-related sets in various moduli. To define the general formula, let us first rewrite the Formula 3.1 using generic symbols to indicate the cyclic collection and remainder sets, and multiset sums instead of set unions:

FORMULA 3.2: $\Phi \uplus x + \Psi \quad \mathcal{Z} \quad \Phi \uplus -x + \Psi$

Formula 3.2 holds true provided that Φ and Ψ satisfy certain conditions. The set Φ is a cyclic collection of cardinality m/ϕ that is generated by the cycle interval ϕ starting from {0}, where each pitch class in the collection is some multiple of ϕ. The

remainder set Ψ is a non-cyclic collection of a particular formation such that the two transpositions (by $\pm x$) of the set yields two identical spanning vectors (using Soderberg's terminology) in respect to the static cyclic collection. The remainder set cannot be just any set; rather, it also must satisfy certain criteria (see below).

Patterson's generalized formula (see Ch. 2, Example 2.5) shows that dyads can be paired with cyclic collections to make z-related sets, so long as the span of the dyad is half of the cycle interval ($m/2$, using his notation). The problem with this explanation is that it accounts only for a small number of z-related pairs—in mod12, only 2 out of 23 z-related pairs. With different criteria for which sets can combine with cyclic collections to make z-related sets, however, we can account for more (if not all) of the Type 1 pseudohomometric sets that have one cyclic-collection subset. The following four criteria, which I propose as an alternative, yield instead the remainder sets for all (known to me) single-cycle z-related pairs that share the same Fourier partition, including all of the single-cycle z-related pairs in mod12 (12 out of 23 z-related pairs in mod12):

Criterion 1. Ψ is of a set class that possesses 'cyclic invariance': the property such that an instance of the set class is able to transform onto another instance of the same set class exclusively by transposition of individual pitch classes (though not all pcs) by the interval $d\phi$ (where d is some integer).

Criterion 2. Ψ does not abstractly include Φ as a subset. (This eliminates possible malformations.)

Criterion 3. Ψ is symmetrical at index 0 in modϕ .

Criterion 4. Ψ is not symmetrical at index 0 or $m/2$ in mod$m$. (Otherwise, the two sides of the formula will be of the same set class.)

If a set satisfies all four criteria (given a certain cyclic collection), then the set will be able to be pair with the cyclic collection to make a valid z-relation formula.[2] The first two criteria yield the set classes that are possible remainder sets for a given cycle, while the other two put constraints on the transposition levels of the various instances of the set classes.

Criterion 1 allows for set classes that are non-symmetrical, so long as they possess cyclic invariance—that is, they can map onto themselves by transposition of some (but not all) pitches by the cycle interval. For example, take the set class [015] in mod12. The set class is not inversionally symmetrical, but it has cyclic invariance in respect to the [06] cycle since the pitch-class set {0, 1, 5} transforms onto another instance of the same set class by transposing certain pitch classes by T_6: transforming to the set {1, 5, 6} by a transposition of the pitch class {0} by T_6, or transforming to the set {7, 11, 0} by a transposition of the pitch classes {1} and {5} by T_6.

Criterion 2 places further limits to which set classes can form valid remainder sets by excluding set classes that possess the cyclic collection Φ as a subset. In many cases, Criterion 2 is not necessary; but in some cases, sets that include Φ fail to make valid z-relation equations. Of the available set classes yielded by first two criteria, Criteria 3 and 4 place restrictions regarding the possible transposition levels. Though, for example, the pitch-class set {1, 2, 6} is of the set class [015], which possesses

[2] Again, I cannot prove this claim, but instead am relying on empirically influenced intuitions.

cyclic invariance in respect to the [06] cycle, an equation with $\Psi = \{1, 2, 6\}$ will not create z-related sets; that is,

$$\{0, 6\} \uplus x + \{1, 2, 6\} \;\mathcal{Z}\; \{0, 6\} \uplus -x + \{1, 2, 6\}$$

is not a true statement. Rather, in order for an instance of the set class [015] to be a valid remainder set, it must satisfy Criteria 3-4: it must be symmetrical at index 0 (mod6), and it must not be symmetrical at index 0 or 6 (mod12). Criteria 3 eliminates invalid formulas such as the one just shown (with {1, 2, 6}), and Criteria 4 prevents the formulas from only producing sets of the same set class. For the set class [015], then, the instances of the set class that makes a valid remainder set (Ψ) in respect to [06] are the pitch-class sets {0, 1, 5}, {6, 7, 11}, {7, 11, 0}, {1, 5, 6}, {3, 4, 8}, {9, 10, 2}, {10, 2, 3} and {4, 8, 9}, each of which could be used to make a z-relation formula.

Table 3.2 lists in the left-hand column all of the set classes in mod12 that satisfy Criteria 1-2, and in the middle column one instance of each set class that satisfies Criteria 3-4. All of the pitch-class sets in the middle column of Table 3.2 can be used as remainder sets with the cyclic collection {0, 6} to make valid z-relation equations. For instance, the Guidonian hexachord (pitch-class set {0, 2, 4, 5, 7, 9}) generates a valid z-relation equation:

$$\{0, 6\} \uplus x + \{0, 2, 4, 5, 7, 9\} \;\mathcal{Z}\; \{0, 6\} \uplus -x + \{0, 2, 4, 5, 7, 9\}$$

Though the formula with the Guidonian hexachord remains true for all real-number values of *x* where *x* is not divisible by 3, all integer values of *x* yield two z-related

multisets with pitch-class duplications. When the cyclic collection and the remainder set overlap pitch classes, then the resultant set must include all copies of the pitch classes, in fact, or else the z-relation will not necessarily hold. As indicated by the asterisks in the right-hand column of the table, many of the set classes in the table only create either real-number (non-integer) z-related pairs or z-related pairs with duplicate copies of certain pitch classes.

Table 3.2: List of sets in mod12 that satisfy the conditions for Ψ, given the cyclic interval $\phi = 6$.

Set classes that satisfy Criteria 1 and 2	Possible positioning of set class according to Criteria 3 and 4.	Only generates multisets with multiple copies of certain elements, or non-integer sets (* for yes, and - for no).
[03]	{0, 3}	-
[024]	{0, 2, 4}	-
[015]	{0, 1, 5}	-
[0124]	{11, 0, 1, 3}	-
[0134]	{11.5, 0.5, 2.5, 3.5}	-
[0125]	{11.5, 0.5, 1.5, 4.5}	-
[0135]	{0, 1, 3, 5}	-
[0235]	{11, 1, 2, 4}	*
[0237]	{10.5, 0.5, 1.5, 5.5}	-
[0247]	{11, 1, 3, 6}	-
[0347]	{10, 1, 2, 5}	*
[0148]	{11, 0, 3, 7}	-
[0358]	{0.5, 3.5, 5.5, 8.5}	-
[01235]	{11, 0, 1, 2, 4}	*
[01245]	{0, 1, 2, 4, 5}	*
[02347]	{10, 0, 1, 2, 5}	*
[02357]	{11, 1, 2, 4, 6}	*
[01358]	{11, 0, 2, 4, 7}	*
[01458]	{0, 1, 4, 5, 8}	*
[012345]	{0, 1, 2, 3, 4, 5}	*
[023457]	{0, 2, 3, 4, 5, 7}	*
[013458]	{0, 1, 3, 4, 5, 8}	*
[024579]	{0, 2, 4, 5, 7, 9}	*
[014589]	{0, 1, 4, 5, 8, 9}	*

Given these four criteria for Ψ, Formula 3.2 produces all of the z-related pairs in mod12 that partition into one cyclic collection. With Formula 3.2 established, let us now turn to look at other possible reformulations of Formula 3.2. In the section ahead, I will associate these various formulations, which I consider to be definitive transformational paths, to particular *z-transformations*, which are contextual transformations that change a z-related set to its z-related partner—contextual in that they affect each set in certain ways depending on the structure of the set. For z-related sets with one cyclic collection, I identify overall two main types of z-transformation: 1) *z-transposition* and *z-inversion*, which derive from Formula 3.2, and 2) what I call the *cyclic-sub z-transformation*, which is based on cyclic invariance.

3.2 Z-transposition and z-inversion

Formula 3.2 describes a relationship between z-related sets where one z-related set maps onto the other by transposing the remainder subset.

$$\Phi \uplus x + \Psi \quad \mathcal{Z} \quad \Phi \uplus -x + \Psi$$

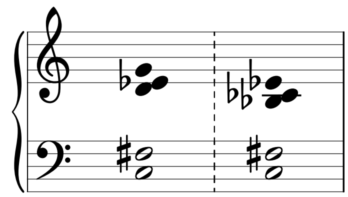
Accordingly, Formula 3.2 reflects what I call *z-transposition* (or Z_T), as each remainder set is equidistantly transposed away from the cyclic collection by some interval x. The formula shows that a z-related set on one side of the formula can be transformed to its z-related partner by transposing the remainder set by $\pm 2x$, depending on which side of the formula the initial z-related set is found. Take for instance the z-related set-class pair [01457] and [01258], which derives from the union of the cyclic collection {0, 6} and the remainder set {0, 1, 5} (see Example

3.1). The z-related pair arises from placing the two remainder sets equidistantly from some point in the cyclic collection.

Example 3.1: An example of z-transposition: set classes [01457] and [01258].

$\{0, 6\} \uplus x + \{0, 1, 5\} \;\; \mathcal{Z} \;\; \{0, 6\} \uplus -x + \{0, 1, 5\}$
let $x = 2$

$\{0, 2, 3, 6, 7\} \;\; \mathcal{Z} \;\; \{10, 11, 0, 3, 6\}$



The distance between the two remainder subsets is $2x$. Thus, for one z-related set to map onto the other, either the remainder subset on the left-hand side of Example 3.1 transposes down by $2x = 4$ to its partner, or the right-hand side transposes up by 4. In other words, z-transposition (Z_T) is the contextual transformation that transposes the remainder set by $\pm 2x$, depending which set of the z-related pair is being transformed. If we let the two sets in Example 3.1 be the sets A and B, then the relationship would be expressed as $A = Z_T(B)$, or, since Z_T is its own inverse, as $B = Z_T(A)$.

We can change the formula so that the remainder set is held invariant instead of the cyclic collection. Since the z-relation is unaffected by transposition or inversion, we can transpose (by adding some positive or negative integer) or invert (by multiplying by –1) either side of the formula without negating the truthfulness of the equation. For example, by transposing the right-hand side of Formula 3.2 by x, the formula is changed so that the cyclic collection on the right-hand side is transposed, instead of the remainder set. Between the two sides, then, neither the cyclic collection nor the remainder set is held invariant; instead, x is added to either the cyclic collection or the remainder set:

$$\Phi \uplus x + \Psi \quad \mathcal{Z} \quad x + (\Phi \uplus -x + \Psi)$$

FORMULA 3.3: $$\Phi \uplus x + \Psi \quad \mathcal{Z} \quad x + \Phi \uplus \Psi$$

If in addition we do the same to the left-hand side, subtracting $-x$, then the formula now keeps the remainder set invariant between the two sides, and the cyclic collection is transposed:

$$-x + (\Phi \uplus x + \Psi) \quad \mathcal{Z} \quad x + \Phi \uplus \Psi$$

FORMULA 3.4: $$-x + \Phi \uplus \Psi \quad \mathcal{Z} \quad x + \Phi \uplus \Psi$$

As one would expect, transposition of the cyclic collection by $-x$ (on the left-hand side in this last formula) produces the same set class as would a transposition of the remainder set by x (like Formula 3.2).

Through a simple modification, we can also reshape the formula so that it expresses the z-relation in terms of inversion instead of transposition. If we multiply the right-hand side of Formula 3.2 by –1 (inverting the set around 0), we get:

$$\Phi \uplus x + \Psi \quad \mathcal{Z} \quad -1 * (\Phi \uplus -x + \Psi)$$

which reduces to,

FORMULA 3.5: $$\Phi \uplus x + \Psi \quad \mathcal{Z} \quad \Phi \uplus x - \Psi$$

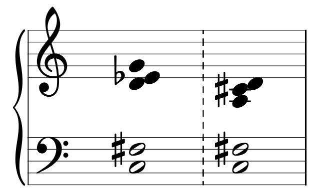
The formula still utilizes the variable x, but now as an axis of inversion. Accordingly, Formula 3.5 describes *z-inversion* (Z_I). As illustrated in Example 3.2, inverting the remainder set around x forms the z-related partner.

Any z-related pair that is expressible as the union of one cyclic collection and one remainder set can be conceived in terms of both z-transposition and z-inversion. If z-transposition is possible, then so will be z-inversion, and vice versa. Since the

Example 3.2: An example of z-inversion: set classes [01457] and [01258].

$\{0, 6\} \uplus x + \{0, 1, 5\} \;\; \mathcal{Z} \;\; \{0, 6\} \uplus x - \{0, 1, 5\}$
let $x = 2$

$\{0, 2, 3, 6, 7\} \;\; \mathcal{Z} \;\; \{6, 9, 0, 1, 2\}$



cyclic collection Φ is always both transpositionally and inversionally symmetrical, we can expand these two transformations to form a group (in the mathematical sense, with closure, associativity, identity and inversibility). For a given z-related pair, the group of transformations yields all of the instances of the two z-related set classes that share the same cyclic-collection pitch-class set. All the instances are related by transposition, inversion, z-transposition, z-inversion, or by some transposition by the cycle interval ϕ of the z-transposition or the z-inversion. The number of pitch-class sets in such a collection depends on the cycle interval ϕ and on whether or not one or both of the resulting sets are inversionally symmetrical (given a certain x)—if both resultant z-related sets are non-symmetrical, then there are $g = 4 * m/\phi$ pitch-class sets that share the same cyclic-collection subset. If one or both of the resultant z-related sets is symmetrical, then there will be fewer unique pitch-class sets (see below). If the collection has $g = 4 * m/\phi$ pitch-class sets, then the group of transformations will also be simply transitive, in that each transformation in the group, Z_T and Z_I (and their transposition by $d\phi$) yields unique pitch-class sets.[3]

[3] The simply transitive group is discussed by Lewin in numerous places (such as his *GMIT*, 1987), and is a central aspect of the GIS. Satyendra 2004, among others, offers a clear and concise of GMIT and the simply transitive group in the context of Lewin's theory.

For now, let us consider the collection in general, writing each instance as some form of Φ, Ψ and x. Each of the sets is either transposed by T_0 ($d = 0$) or transposed by some multiple ($0 < d <$ cardinality of Φ) of the cycle interval ϕ:

FORMULA 3.6:

$$\Phi \uplus \begin{array}{l} x + \Psi \\ -x - \Psi \\ d\phi + (x + \Psi) \\ d\phi + (-x - \Psi) \end{array} \qquad \mathcal{Z} \qquad \Phi \uplus \begin{array}{l} -x + \Psi \\ x - \Psi \\ d\phi + (-x + \Psi) \\ d\phi + (x - \Psi) \end{array}$$

The various pitch-class sets on one side of Formula 3.6 (Φ plus one remainder set) are all z-related to the pitch-class sets on the other side by Z_T, by Z_I, or by a transposition (by some multiple of ϕ) of a Z_T or Z_I.

As an example, let us reconsider one of the pairs of cyclic-collection and remainder subsets from above: the remainder set {0, 1, 5} and cyclic collection {0, 6} in mod12 (see Example 3.3). Since the cycle interval ϕ is 6, and since {0, 1, 5} is not symmetrical, there are 4 * 12/6 = 8 pitch-class sets that all share the same cyclic collection, with 4 instances of each of the two z-related set classes. The 8 pitch-class sets are shown in Example 3.3 with the variable $x = 2$, forming 8 instances (4+4) of [01457] and [01258]. Among these 8 pitch-class sets, the {0, 6} cyclic collection remains invariant while only the remainder set is transformed. Among the 4 pitch-class sets on each of the two sides, the sets are all related by multiplication by -1 and/or T_6, while between the two sides each pitch-class set is related to a pitch-class set on the other side by z-transposition, by z-inversion, or by a T_6 transposition of one of those two.

Example 3.3: Instances of z-related pair that share the same cyclic collection, where $\Phi = \{0, 6\}$, $\Psi = \{0, 1, 5\}$ and $x = 2$.

$$\{0, 6\} \uplus \begin{array}{l} x + \{0, 1, 5\} \\ -x - \{0, 1, 5\} \\ 6 + (x + \{0, 1, 5\}) \\ 6 + (-x - \{0, 1, 5\}) \end{array} \qquad \mathcal{Z} \qquad \{0, 6\} \uplus \begin{array}{l} -x + \{0, 1, 5\} \\ x - \{0, 1, 5\} \\ 6 + (-x + \{0, 1, 5\}) \\ 6 + (-x + \{0, 1, 5\}) \end{array}$$

<u>let $x = 2$</u>

$$\{0, 6\} \cup \begin{array}{l} \{2, 3, 7\} \\ \{10, 9, 5\} \\ \{8, 9, 1\} \\ \{4, 3, 11\} \end{array} \qquad \mathcal{Z} \qquad \{0, 6\} \cup \begin{array}{l} \{10, 11, 3\} \\ \{2, 1, 9\} \\ \{4, 5, 9\} \\ \{8, 7, 3\} \end{array}$$

Starting from one of the pitch-class sets in the collection, all of the other sets are related by I, Z_T, Z_I, or a T_6 transposition of those. If, for example, we begin with the set on the top left of Example 3.3, the set $\{0, 6\} \cup \{2, 3, 7\}$, then the other sets would be related by the following transformations (X = identity):

X	Z_T
I	Z_I
T_6	T_6Z_T
T_6I	T_6Z_I

Instead, if we start from the second pitch-class set from the top right, the other sets are related by the following transformations:

Z_I	I
Z_T	X
T_6Z_I	T_6I
T_6Z_T	T_6

All z-related pairs that can be expressed by z-transposition/z-inversion participate in a group structure in the form of Formula 3.6. However, if the remainder set is symmetrical, certain values of x will result in one or both of the z-related sets being symmetrical. If the resultant z-related sets are symmetrical, then the number of unique pitch-class sets in the collection decreases—that is, if one resultant z-related set is symmetrical, then $g = 3 * m/\phi$, and if both z-related sets are symmetrical, then $g = 2 * m/\phi$. For instance, in the case where $\Phi = \{0, 6\}$ and $\Psi = \{0, 2, 4\}$ and $x = 1$ (mod12) (see Example 3.4), one of the resultant z-related sets is symmetrical, [01356] (the left-hand side of Example 3.4), and thus there are only $g = 3 * 12/6 = 6$ distinct pitch-class sets in the collection.

Example 3.4: Group with fewer pitch-class sets, where $\Phi = \{0, 6\}$, $\Psi = \{0, 2, 4\}$ and $x = 1$.

$$\{0, 6\} \uplus \begin{array}{l} x + \{0, 2, 4\} \\ -x - \{0, 2, 4\} \\ 6 + (x + \{0, 2, 4\}) \\ 6 + (-x - \{0, 2, 4\}) \end{array} \quad \mathcal{Z} \quad \{0, 6\} \uplus \begin{array}{l} -x + \{0, 2, 4\} \\ x - \{0, 2, 4\} \\ 6 + (-x + \{0, 2, 4\}) \\ 6 + (-x + \{0, 2, 4\}) \end{array}$$

<u>let $x = 1$</u>

$$\{0, 6\} \cup \begin{array}{l} \{1, 3, 5\} \\ \{7, 9, 11\} \\ \{7, 9, 11\} \\ \{1, 3, 5\} \end{array} \quad \mathcal{Z} \quad \Phi \cup \begin{array}{l} \{11, 1, 3\} \\ \{9, 11, 1\} \\ \{5, 7, 9\} \\ \{3, 5, 7\} \end{array}$$

For this z-related pair (in Ex. 3.4), there are then two transformations of the remainder set that are equivalent when $x = 1$:

$$\begin{array}{lll} 1 + \{0, 2, 4\} & = 6 + (-1 - \{0, 2, 4\}) & = \{1, 3, 5\} \\ -1 - \{0, 2, 4\} & = 6 + (1 + \{0, 2, 4\}) & = \{7, 9, 11\} \end{array}$$

Thus, starting from one of the pitch-class sets on the right-hand side of Ex. 3.4, certain z-transformations will result in the same z-related pitch-class set; in other words, the group of transformations is no longer simply transitive. Nevertheless, though the z-transformations are no longer "one-to-one" with such sets, it remains true that for any z-related set that partitions into one cyclic collection and a remainder set, there are will always be 2 * m/ϕ possible z-transformations, regardless of whether or not each of these z-transformations yields a unique pitch-class set.

3.3 Cyclic-sub z-transformation

In addition to the two basic z-transformations presented thus far, z-transposition and z-inversion, there is a third z-transformation. Technically speaking, this third z-transformation is an instance of one of the transformations in the Z_T/Z_I group, and thus one could argue that it is not a legitimately distinct z-transformation. However, because of its conceptual importance I prefer to give this third z-transformation more weight. The z-transformation is based on a notion introduced in Criteria 1 for Ψ sets: cyclic invariance. That remainder sets possess cyclic variance—the set-class property whereby an instance of a set class can map onto another instance of the same set class by transposing only some pitch classes by $d\phi$ (some multiple of the cycle interval)—is a key feature of Type 1 pseudohomometric sets that have one cycle-collection subset, in that the transformation of the remainder set to another set of the same set class by way of a partial transposition by $d\phi$ almost always transforms the z-related set to its z-partner (see below for an explanation of when it fails). That is,

> Given that A is a valid remainder set for ϕ, and that A and B are of the same set class, if A can transform onto B by transposing some (but not all) pitch-classes by $d\phi$ (partial transposition by $d\phi$), then $\Phi \uplus$ A and $\Phi \uplus$ B are either z-related or (in some cases) of the same set class.

I will refer to this z-transformation as the *cyclic-sub z-transformation*, or Z_{CS}. When considering one cycle in particular, it may be possible to refer to it as, for example, the tritone-sub z-transformation, or even the major-third-sub z-transformation. However, in order to avoid any confusion with the term 'tritone-sub', which in Jazz theory takes a slightly different definition, I have decided to use the term 'cyclic-sub' throughout.

To develop a z-relation formula that uses the cyclic-sub z-transformation, let us first consider some set in the form of:

$$\Phi \uplus x + \Psi$$

with any given real number x not divisible by $\phi/2$. Given the pitch-class set, let us now consider a pitch-class set that shares the same cyclic collection Φ, but that has a remainder set Ψ that is a cyclic-sub of the original Ψ. This second pitch-class set is either z-related to the first, or in a few cases, of the same set class as the first. Using the symbol CS to indicate the cyclic-sub function—the function that changes an instance of a set class to another instance of the same set class by transposing certain pitch classes by ϕ—we write the expression as the following:

FORMULA 3.7: $\Phi \uplus x + \Psi \;\mathcal{Z}\; \Phi \uplus x + \mathrm{CS}(\Psi)$

Since CS contextually affects the remainder set, the following is also true:

$$\Phi \uplus x + \Psi \;\mathcal{Z}\; \Phi \uplus \mathrm{CS}(x + \Psi)$$

If we let the two sets in the formula be the sets A and B, then the z-transformation is expressed as $A = Z_{CS}(B)$ or as $B = Z_{CS}(A)$. The formula very efficiently generates the z-related partner, making it the most practical of all the z-transformations. Given some random z-related set with one cyclic-collection subset, the z-related partner can be found simply by determining the possible cyclic-subs of the remainder set. The variable x plays no role in the transformation, other than describing the relationship between the remainder set and the cyclic collection in the original set, and thus it is not even necessary to know the value of x to calculate the cyclic-sub z-transformation.

Let us consider an example of the cyclic-sub z-transformation, using the example z-related pair from above, where $\Phi = \{0, 6\}$ and $\Psi = \{0, 1, 5\}$. The remainder set Ψ is able to transform onto another instance of the same set class by transposing by T_6 either pitch class $\{0\}$ or pitch classes $\{1, 5\}$. There are two possible z-transformations, and thus to gain a little more precision with our notation, let us write the function as $CS_{\{pcs\}}$ to indicate which pitch classes are affected by the cyclic-sub. In this case, with the remainder set $\{0, 1, 5\}$, the possible cyclic-subs are $CS_{\{0\}}\{0, 1, 5\}$ and $CS_{\{1, 5\}}\{0, 1, 5\}$. Accordingly, the following two z-relation equations are true:

$$\{0, 6\} \uplus x + \{0, 1, 5\} \;\; \mathcal{Z} \;\; \{0, 6\} \uplus x + CS_{\{0\}}\{0, 1, 5\}$$

$$\{0, 6\} \uplus x + \{0, 1, 5\} \;\; \mathcal{Z} \;\; \{0, 6\} \uplus x + CS_{\{1, 5\}}\{0, 1, 5\}$$

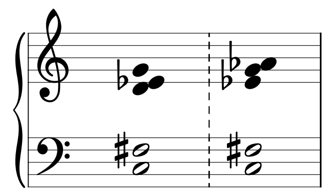
For example, let us say that x is 2, in which case our starting set is $\{0, 2, 3, 6, 7\}$. The first cyclic-sub ($CS_{\{0\}}$) affects the pitch class $\{2\}$ of the remainder set $\{2, 3, 7\}$, and

thus the z-transformation based on that cyclic-sub (Z_{CS}) yields the set $\{0, 3, 6, 7, 8\}$, which is z-related to the starting set (see Example 3.5). In a similar manner, the second cyclic-sub z-transformation (using $CS_{\{1, 5\}}$) yields the set $\{0, 1, 2, 6, 9\}$, which is also z-related to the starting set.

Example 3.5: An example of cyclic substitution: set classes [01457] and [01258].

$\{0, 6\} \uplus x + \{0, 1, 5\} \;\; \mathcal{Z} \;\; \{0, 6\} \uplus x + CS_{\{0\}}\{0, 1, 5\}$
let $x = 2$

$\{0, 2, 3, 6, 7\} \;\; \mathcal{Z} \;\; \{0, 3, 6, 7, 8\}$



By definition, all valid remainder sets Ψ are able to map via cyclic substitution onto some other instance of the same set class. As just mentioned, the set $\{0, 1, 5\}$ has precisely two sets that are related by cyclic substitution. Some other sets can map onto more than two other instances of the same set class, such as the set [024] (see Example 3.6).

Example 3.6: Possible cyclic-subs for $\{0, 2, 4\}$, where $\phi = 6$ (mod12).

$CS_{\{0\}}\{0, 2, 4\} = \{2, 4, 6\}$
$CS_{\{4\}}\{0, 2, 4\} = \{10, 0, 2\}$
$CS_{\{0, 2\}}\{0, 2, 4\} = \{4, 6, 8\}$
$CS_{\{2, 4\}}\{0, 2, 4\} = \{8, 10, 2\}$

The remainder set [024] not only has four different cyclic-subs, but it is also one of few remainder sets that under the cyclic-sub z-transformation forms sets of the same set class (for certain values of *x*). As noted, the cyclic-sub z-transformation yields z-related sets or sets of the same set class. For the most part, it forms z-related sets; however, in a few cases, with certain remainder sets and with certain values of *x*, the

cyclic-sub z-transformation fails to generate z-related sets, but forms instead sets of the same set class. With the remainder set [024] (where ϕ=6 mod12), we can see that there are cases where the cyclic-sub z-transformation fails (here, $\cong$ indicates T_nI equivalence):

$$\{0, 6\} \cup -1 + \{0, 2, 4\} \cong \{0, 6\} \cup -1 + CS_{\{4\}}(\{0, 2, 4\})$$
$$\{11, 0, 1, 3, 6\} \cong \{6, 9, 11, 0, 1\}$$

$$\{0, 6\} \cup 5 + \{0, 2, 4\} \cong \{0, 6\} \cup 5 + CS_{\{4\}}(\{0, 2, 4\})$$
$$\{5, 6, 7, 9, 0\} \cong \{0, 3, 5, 6, 7\}$$

…

Instead of being z-related, the two sides of the equations are of the same set class. Such cases seem to occur (judging without a proof) when the number of possible cyclic-subs of the remainder set does not evenly divide into g (=2|3|4 * 12/ϕ). In the case where Ψ = {0, 2, 4} and x = 1, the number of possible cyclic-subs (4) does not evenly divide into g (= 3 * 2 = 6). However, since g is dependent on x, changing the value of x changes whether or not all the cyclic-sub z-transformations yield z-related sets. For the remainder set {0, 2, 4}, if we changed x to something like π, for example, then g would be 8, and all of the cyclic-sub z-transformations would yield z-related sets.

As mentioned before, the cyclic-sub z-transformation is not a unique transformation, but instead is always equivalent to some z-transposition, z-inversion, or $T_{d\phi}$ transposition of the z-transposition or z-inversion. In the case given above where Ψ = {0, 1, 5} and x = 2, $Z_{CS\{0\}}$ produces the same pitch-class set as would a T_6 transposition of the z-inversion (the set {0, 3, 6, 7, 8}), and $Z_{CS\{1, 5\}}$ produces the

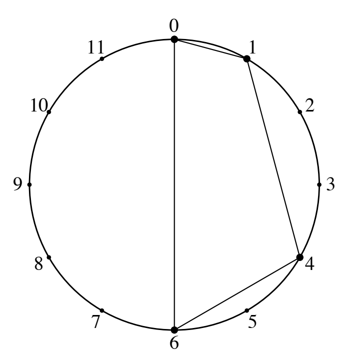
same pitch-class set as would the z-inversion (the set {0, 3, 6, 7, 8}). Nevertheless, there are analytical cases in which Z_{CS} may be the preferred z-transformation. In Example 3.5, for example, we can see that the two vertical sonorities are identical, except that the first chord has a D and the second an A♭, which are a tritone apart. In cases as clear as Ex. 3.5 it is certainly convenient to visualize the transformation as a cyclic-sub instead of a transposed z-inversion. However, in other cases, with different voice leading, it may be better to understand the z-transformation as a transposed z-inversion. That is to say, though the cyclic-sub duplicates explanations given previously, this multiplicity in explanation is arguably beneficial in regards to music analysis, in that it provides another means for interpreting relationships between sets that are z-related.

3.4 Orthogonality and the cyclic-sub z-transformation

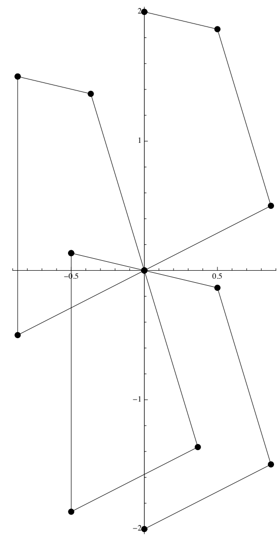
Further to being an alternate transformation, the cyclic-sub z-transformation reveals how the z-relation is an orthogonal (perpendicular) relationship. Before, the z-relation has been described in terms of positive and negative, where one side of a z-relation formula uses a positive sign, and the other side the negative sign. As I shall show, the z-relation, however, is also a 'perpendicular' relationship. By adapting a method drawn from crystallography for graphing sets, I will show that the cyclic-sub z-transformation in fact works because it plays upon this perpendicularity of the z-relation. The method involves what Martin J. Buerger (1950 and 1959) calls 'vector graphs', which plot the intervals within a set starting from the center of an axis.

Before showing the relationship between orthogonality and the cyclic-sub z-transformation, let us take a moment to flesh out the vector graph. Since Buerger's vector graphs are modeled for sets in 2 or more dimensions, I will modify his method as to handle sets in a 1-dimensional periodic space. To make a vector graph of a set, let us start with the regular graph of a pitch-class set on the unit circle, using the pitch-class set {0, 1, 4, 6} (see Example 3.7). Next, let us duplicate the graph on the unit circle, so that there are as many graphs of the set as there are pitch classes in the set. In this case, there are 4 pitch classes, so we will need 4 identical graphs of the set. Next, arrange the four graphs, without rotating them, so that each one of the graphs places a unique pitch class at the center of another larger axis. Example 3.8 illustrates this. As can be seen, one graph is situated so that pitch class {0} is at the center of the large axis, while another has {1} at the center, and so on. Next, strip away the lines in the graphs, leaving behind only the points. Then, starting from the center of the axis, draw lines to each of the different points (see Example 3.9). This final graph is the vector graph.

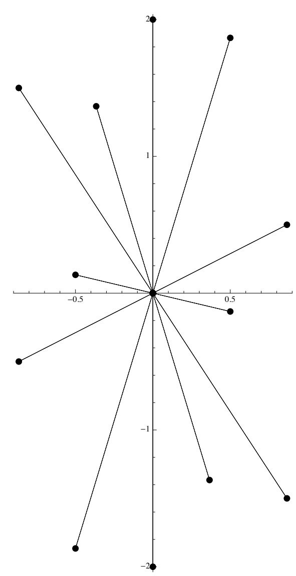
Example 3.7: Graph of {0, 1, 4, 6} on the mod12 unit circle.



Example 3.8: Four pc-set graphs of {0, 1, 4, 6} arranged on another larger axis so that each pitch class is placed at the center of a large axis.



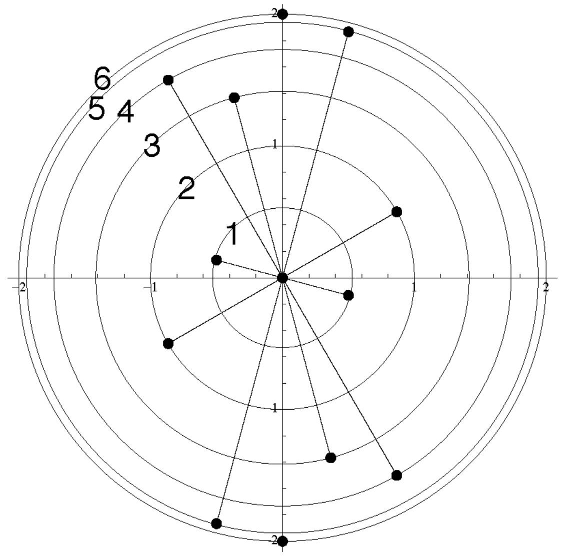
Example 3.9: Vector graph of {0, 1, 4, 6} (mod12), with lines from the center of the axis to each point in the graph.



The vector graph is essentially a graph of all of the interval classes within a given set, which are represented as vectors (i.e., lines with a particular magnitude and direction). However, the vector graph tallies directed intervals, and so each interval

class is graphed twice, once in each direction (0→1 vs. 1→0, etc.). The number of points that lie at particular distances from the center of the axis shows how many of each of the interval classes are in the set. In the graph in Ex. 3.9, there are only $m/2$ = 6 distances between the points and the center of the axis, spread unequally from the center to the outer distance. To show this, Example 3.10 adds rings to the vector graph, which indicate each of the six interval classes.

Example 3.10: Vector graph of {0, 1, 4, 6} (mod12) with added interval-class rings.



As can be seen, there are two points on each of the rings in Ex. 3.10, indicating that within the set {0, 1, 4, 6} there are two of each of the 6 interval classes (again, counting twice for each interval class since it considers direction). Though they might not be seen in the vector graphs in Exs. 3.9 and 3.10, there are also four points at the center of the axis. Since by making the vector graph we placed each of the pitch classes of the set in the center of the axis, it is clear that these four points at the center of the graph indicate the cardinality of the set. However, the four points also indicate

4 instances of ic0, judging by their distance from the center of the axis. Thus, an interval array taken from the vector graph in Exs. 3.9 and 3.10 would be:

interval array: 4 2 2 2 2 2 2

In this way, the vector graph yields the same interval array as would a T-matrix, except that it counts (directed) interval classes, as opposed to the ordered intervals of the T-matrix:[4]

	0	1	4	6
0	0	1	4	6
11	11	0	3	5
8	8	9	0	2
6	6	7	10	0

interval array: 4 1 1 1 1 1 2 1 1 1 1 1

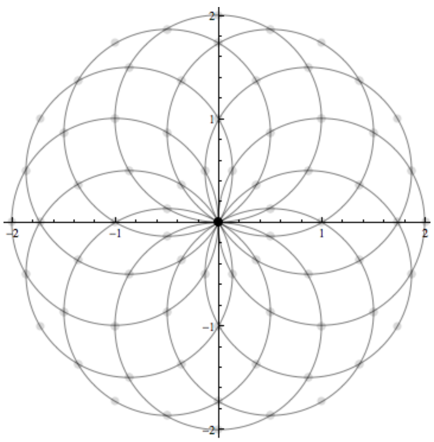
Not only do the T-matrix and vector graph both produce an array of all the directed intervals within a set (albeit the vector graph showing them as interval classes), but they also both have a 0-th entry, which is the cardinality of set. However, the vector graph shows something that the T-matrix does not, in that it also shows the direction of the intervals within the unit circles.

The vector graph lies within what Buerger calls 'vector space', which is the space that consists of all of the possible intervals in all possible directions. For the case of mod12 periodic space, the vector space is as shown in Example 3.11. Graph 3.11a shows that that the space derives from 12 pitch-class circles; 3.11b shows only the end points of the intervals; and graph 3.11c add interval-class rings to 3.11b.

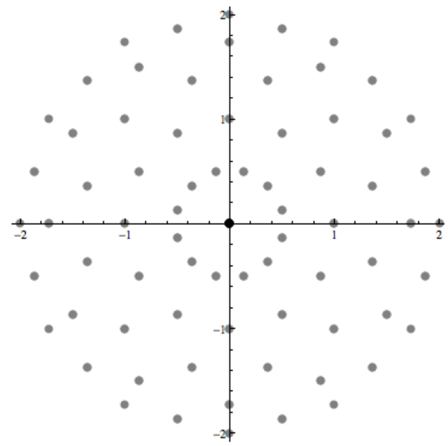
[4] The interval array generated by the T-matrix is also identical to the output of autocorrelation, as used in signal processing.

Example 3.11: Vector space for mod12, shown as three different graphs.

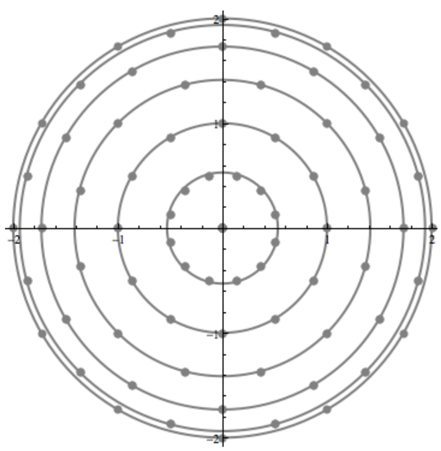
a)



b)



c)

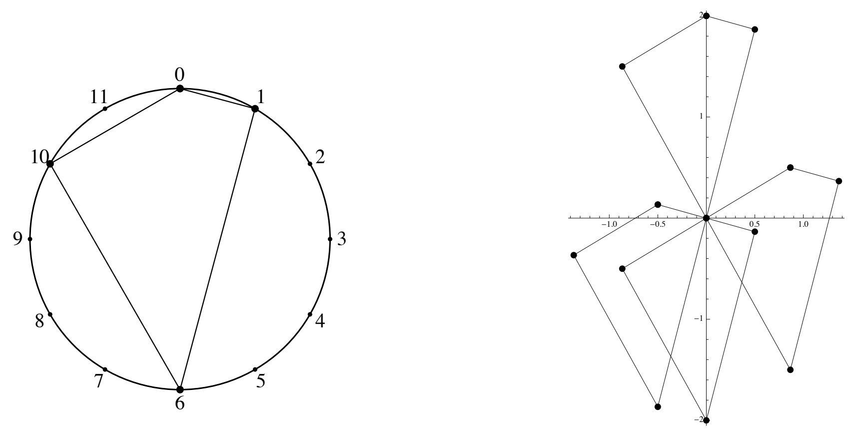


Let us now consider the vector graphs of a pair of z-related sets to see how the cyclic-sub z-transformation plays on the orthogonality of the z-relation. We already have a vector graph for {0, 1, 4, 6}, so let us make a vector graph for the z-related pair (set class [0137]), making sure, however, to choose a set that is related by cyclic-sub. The set to be graphed will be:

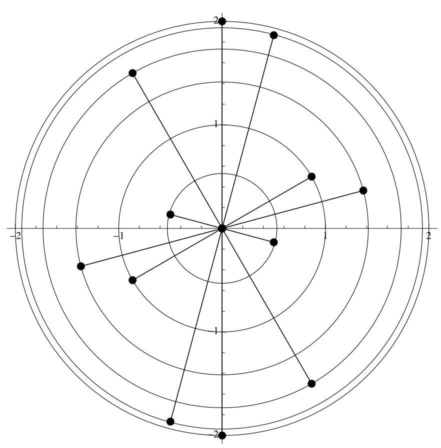
$$\{0, 6\} \cup \{1, 4\} \ \mathcal{Z} \ \{0, 6\} \cup CS_{\{4\}}\{1, 4\}$$
$$\mathcal{Z} \ \{0, 6\} \cup \{10, 1\}$$
$$\mathcal{Z} \ \{10, 0, 1, 6\}$$

Example 3.12 shows the set {10, 0, 1, 6} graphed on the unit circle, and the duplications of the graph placed on a larger axis. Example 3.13 shows the vector graph of the set, with the added interval-class rings.

Example 3.12: The {10, 0, 1, 6} graphed on the unit circle (mod12), and duplicated graphs placed on larger axis.



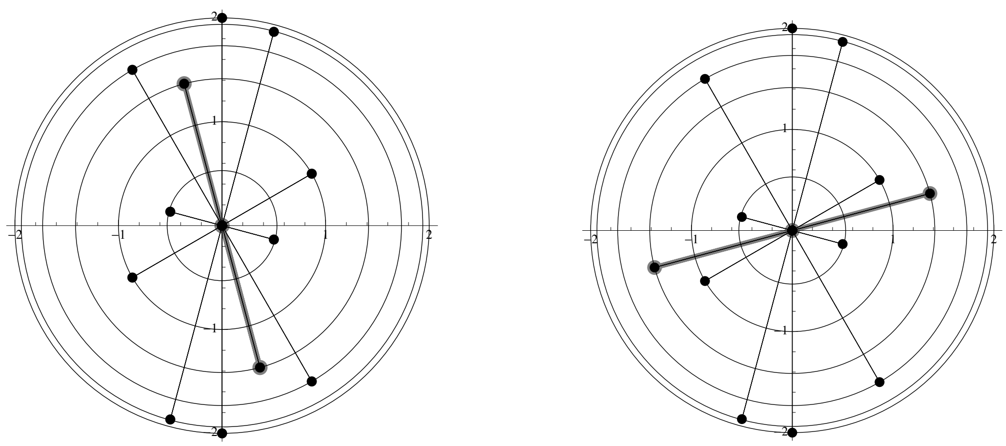
Example 3.13: The vector graph for the set {10, 0, 1, 6} (mod12), with interval-class rings.



Naturally, in comparing the vector graphs of the two sets (Exs. 3.10 and 3.13, shown together in Ex. 3.14), it is expected that the two vector graphs have the same number of vectors (the lines from the center of the axis) for each of the different vector lengths, since the two sets have the same interval-class content. However, judging by the directions of the vectors, we can see that between the two sets all of the vectors are the same except one. In Example 3.14, the two exceptional vectors are marked in bold, which, as can be seen, are perpendicular.

Example 3.14: The vector graphs for {0, 1, 4, 6} and {10, 0, 1, 6}, with non-identical vectors marked in bold.

{0, 1, 4, 6} {10, 0, 1, 6}



It is in this sense that the z-relation is an orthogonal (perpendicular) relationship. However, between the two graphs, not all of the vectors are perpendicular; rather, almost all are identical, while only one is perpendicular. We could change the number of perpendicularly related vectors by transposing one of the sets by T_3, in which case all the vectors would be perpendicular except one. Nonetheless, as long as the two sets are of different set classes, it is impossible that

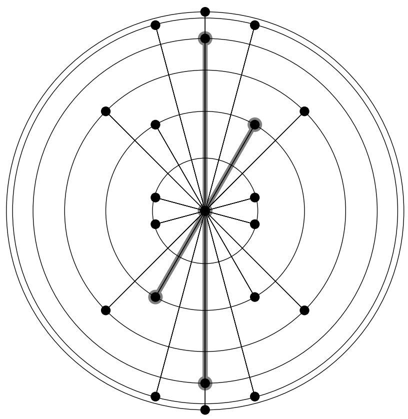
the vectors of the two sets are all perpendicular or all identical. In this way, one could argue that it is better to recognize the relationship as ‘partially orthogonal’, like the ‘partial transposition’ of the algebraic formulas. Regardless of whether we consider this to be an orthogonal relationship or partially orthogonal relationship, the vector graph reveals that for Type 1 pseudohomometric sets with one cyclic collection there are (at least) two instances that have intervals that are all some combination of identical and perpendicular. In addition, the vector graph shows that the phenomenon arises via the cyclic-sub z-transformation (Z_{CS}).

Examples 3.15 and 3.16 give two more examples of vector graphs of z-related sets, marking the perpendicular vectors in bold. As before, the sets in the examples are situated so that they are related by the cyclic-sub z-transformation. Example 3.15 is particularly interesting, in that interval classes 2 and 4 flip, so that ic2 is perpendicular to ic4, and vice versa. I cannot give a definitive explanation for this behavior; nevertheless, what seems to be important is that the two graphs are partially orthogonal on the whole, and not vector by vector.

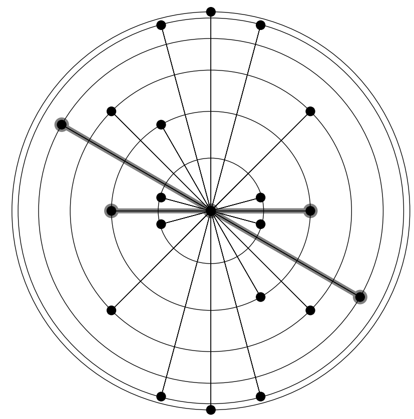
The three z-transformations presented thus far—z-transposition, z-inversion and the cyclic-sub z-transformation—are the core of the present theory. However, in order to expand the scope of the theory to be able to account for z-related sets with more than one cycle, we will now turn to investigate a means for expanding the basic z-transposition/z-inversion formula. As shown by Joseph O’Rourke, Perouz Taslakian and Godfried Toussaint (2008), it is possible to inject pitch classes into already known z-related pairs to create larger cardinality z-related pairs, through a process

Example 3.15: Vector graphs of z-related sets, {0, 1, 3, 5, 6} and {11, 0, 1, 3, 6}.

{0, 1, 3, 5, 6}

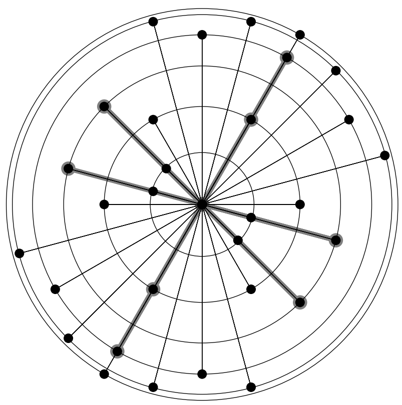


{11, 0, 1, 3, 6}

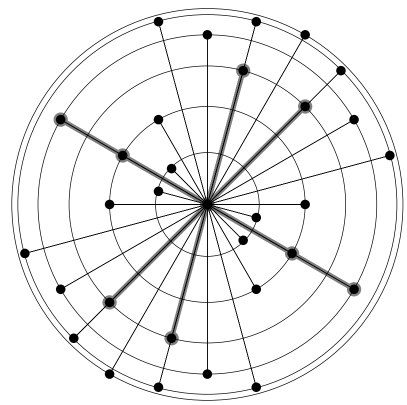


Example 3.16: Vector graphs of z-related sets, {0, 1, 3, 5, 7, 8} and {0, 1, 2, 5, 7, 9}.

{0, 1, 3, 5, 7, 8}



{0, 1, 2, 5, 7, 9}



they call ‘pumping’. Using their ‘pumping lemma’ as a model, I develop a number of original pumping algorithms that can be applied to any z-relation formula in form of Formulas 3.2 and 3.5. The pumping algorithms greatly expand the number of z-related pairs covered by the formulas, extending the formulas to sets with more than

one cyclic-collection subset. In addition, pumping reveals connections between z-related pairs of different cardinalities, showing that there conceivably exists a subset/superset web among the various cardinalities of z-related sets.

3.5 Pumping

Unlike with the pumping lemma given by O'Rourke, et al., I define pumping algorithms in terms of the z-relation formulas. Each pumping algorithm begins with a 'starting' formula in the form of Formulas 3.2 and 3.5. For the sake of clarity, the following presentation only uses Formula 3.2 (the z-transposition formula) as the starting formula, though any of the algorithms can be modified to start with the z-inversion formula by multiplying either side of pumped formula by –1. Also, the algorithms are presented using abbreviated formulas, with the pitch-class sets α and β. Given Φ and Ψ, let α and β be the two sides of a z-transposition formula; that is,

$$\alpha = \Phi \uplus x + \Psi$$
$$\beta = \Phi \uplus -x + \Psi$$

Algorithms 1-4 add pitch classes to α and β according to certain guidelines. The last algorithm (*p*5) begins with a different formula, starting instead with a formula where to the two sides of the formula are equal.

Let us begin with pumping algorithm 1a (shortened as *p*1a), which is perhaps the simplest of all the algorithms. Algorithm *p*1a pumps a z-related pair by adding additional cyclic collections Φ. Like the remainder set Ψ, the added cyclic collections participate in the z-transposition or z-inversion: if the two sides of the formula are related by z-transposition, then the added cyclic collections must also be related by z-

transposition; if the two sides are related by z-inversion, then so must the added cyclic collections. The distance between the added cyclic collections and the static cyclic collection, however, does not have to be *x*, but can be any real number (*y*). There is no limit to how many cyclic collections are added; however, if the resulting pitch-class sets have multiple copies of certain pitch classes (from the multiset sum), then they must be understood as multisets for the z-relation to hold true.

Pumping algorithm 1a:

Let $Y \subseteq \mathbb{R}$ be a chosen set of real numbers. For each $y_n \in Y$, add $y_n + \Phi$ to one side of the formula, and $-y_n + \Phi$ to the other.

$$\alpha \biguplus_{k=0}^{n} y_k + \Phi \quad \mathcal{Z} \quad \beta \biguplus_{k=0}^{n} -y_k + \Phi$$

To get a better idea of how the algorithm works, let us take a look at a couple of examples of z-related pairs pumped by algorithm *p*1a. Take, for instance, a starting formula that has $\Phi = \{0, 6\}$ and $\Psi = \{0, 1, 5\}$ (mod12). Given the starting formula, choose some values for *x* and *Y*—say, $x = 2$, and $Y = \{5\}$. According to algorithm *p*1a, then, the following two sets are z-related:

$$\{0, 6\} \uplus 2 + \{0, 1, 5\} \uplus 5 + \{0, 6\} \;\; \mathcal{Z} \;\; \{0, 6\} \uplus -2 + \{0, 1, 5\} \uplus -5 + \{0, 6\}$$

$$\{11, 0, 2, 3, 5, 6, 7\} \;\; \mathcal{Z} \;\; \{10, 11, 0, 1, 3, 5, 6\}$$

The resultant z-related pair has 2 cyclic-collection subsets, and yet through pumping it is expressible as an algebraic formula. With this particular example (with $x = 2$ and $Y = \{5\}$), the additional [06] subset transforms the starting pair (where $x = 2$) to the heptachordal pair of z-related sets that is the complement of the original pair. Pumping, however, does not always generate complementary pairs, depending on the

values of x and Y. For example, let us change the variables, letting instead $x = 1$ and $Y = \{1, -2\}$, so that two additional Φ sets are added. According to the algorithm,

$$\{0, 6\} \uplus 1 + \{0, 1, 5\} \uplus 1 + \{0, 6\} \uplus -2 + \{0, 6\} \;\; \mathcal{Z} \;\; \{0, 6\} \uplus -1 + \{0, 1, 5\} \uplus -1 + \{0, 6\} \uplus 2 + \{0, 6\}$$

$$\{10, 0, 1, 1, 2, 4, 6, 6, 7\} \;\; \mathcal{Z} \;\; \{11, 11, 0, 0, 2, 4, 5, 6, 8\}$$

In this latter example, we can see that the resultant pair of z-related sets are multisets with multiple instances of certain pitch classes. These two sets are z-related, and they have the same interval content (including i0); however, since they are multisets with multiple copies of certain pitch classes, it is impossible to find a complementation relationship between the pumped pair and the original pair.

Algorithm $p1$a shows that cyclic collections can be added ad infinitum to any given formula without disrupting the z-relation, provided that the added cyclic collections participate in the z-transposition/z-inversion structure of the formula. However, for algorithm $p1$a, the added cyclic collections must be identical to Φ. If we are to add cyclic collections that are of a different set class than Φ, then the algorithm must be adapted slightly. That is, for the z-relation to hold when cyclic collections of a different set class are added, then the transposition/inversion levels of the cyclic collections must be dependent on x, as described by pumping algorithm $p1$b.

Pumping algorithm 1b:

> Let $Y \subseteq \mathbb{R}$ be a chosen set of real numbers, and Φ′ be some other cyclic collection of a different set class than Φ. For each $y_n \in Y$, add $x + (y_n + \Phi')$ to α, and $-x + (y_n + \Phi')$ to β.

$$\alpha \biguplus_{k=0}^{n} x + (y_k + \Phi') \quad \mathcal{Z} \quad \beta \biguplus_{k=0}^{n} -x + (y_k + \Phi')$$

In algorithm $p1$b, the positioning of the added Φ' sets is dependent on x. For example, take the starting formula where $\Phi = \{0, 4, 8\}$ and $\Psi = \{0, 2\}$ (mod12), which given $x = 1$ (or any other odd integer) produces the z-related pair [01348] and [03458]. By pumping the formula with $p1$b, the pair can be transformed to the heptachordal complementary pair; that is, let $\Phi' = \{0, 6\}$ and $Y = \{4\}$:

$$\{0, 4, 8\} \uplus 1 + \{0, 2\} \uplus 1 + (4+\{0, 6\}) \;\mathcal{Z}\; \{0, 4, 8\} \uplus -1 + \{0, 2\} \uplus -1 + (4+\{0, 6\})$$
$$\{0, 4, 8\} \uplus \{1, 3\} \uplus \{5, 11\} \;\mathcal{Z}\; \{0, 4, 8\} \uplus \{11, 1\} \uplus \{3, 9\}$$
$$\{11, 0, 1, 3, 4, 5, 8\} \;\mathcal{Z}\; \{8, 9, 11, 0, 1, 3, 4\}$$

The resulting pumped z-related pair is both a superset of the original pair—the original pair being α and β, where $x = 1$—and the abstract complement of the original pair.

Algorithms 1a and 1b can be combined, so that they are both applied within the same formula. For instance, the following is true:

$$\{0, 4, 8\} \uplus 1 + \{0, 2\} \uplus 3 + \{0, 4, 8\} \uplus 1+(4+\{0, 6\}) \;\mathcal{Z}\; \{0, 4, 8\} \uplus -1 + \{0, 2\} \uplus -3 + \{0, 4, 8\} \uplus -1+(4+\{0, 6\})$$
$$\{0, 4, 8\} \uplus \{1, 3\} \uplus \{3, 7, 11\} \uplus \{5, 11\} \;\mathcal{Z}\; \{0, 4, 8\} \uplus \{11, 1\} \uplus \{1, 5, 9\} \uplus \{3, 9\}$$
$$\{11, 11, 0, 1, 3, 3, 4, 5, 7, 8\} \;\mathcal{Z}\; \{8, 9, 9, 11, 0, 1, 1, 3, 4, 5\}$$

In mod12, there are no z-related pairs without pitch-class duplications that result from mixing $p1$a and $p1$b; however, in larger cardinalities, like mod24, there are many z-related pairs that derive from mixing the two algorithms.

Pumping algorithm $p2$ is like algorithm $p1$a in structure, in that it adds sets that are displaced by intervals Y. However, instead of adding additional Φ sets, it adds additional Ψ sets. Like with $p1$, there is no limit to how many Ψ sets are pumped into

the starting pair with $p2$. Furthermore, algorithm $p2$ can mix with $p1$a, but not with $p1$b.

Pumping algorithm 2:

> Let $Y \subseteq \mathbb{R}$ be a chosen set of real numbers. For each $y_n \in Y$, add $y_n + \Psi$ to one side of the formula, and $-y_n + \Psi$ to the other.
>
> $$\alpha \biguplus_{k=0}^{n} y_k + \Psi \quad \mathcal{Z} \quad \beta \biguplus_{k=0}^{n} -y_k + \Psi$$

For example, let the starting formula be that which has $\Phi = \{0, 6\}$ and $\Psi = \{0, 3\}$ (mod12)—i.e., the formula that produces the all-interval tetrachords, [0146] and [0137], when x is an integer not divisible by 3. Given the variables $x = 1$ and $Y = \{2\}$, then $p2$ yields the hexachordal z-related pair, [012456] and [012348].

$$\{0, 6\} \uplus 1 + \{0, 3\} \uplus 2 + \{0, 3\} \;\; \mathcal{Z} \;\; \{0, 6\} \uplus -1 + \{0, 3\} \uplus -2 + \{0, 3\}$$

$$\{0, 1, 2, 4, 5, 6\} \;\; \mathcal{Z} \;\; \{10, 11, 0, 1, 2, 6\}$$

The hexachordal pair can thus be expressed in two ways: as a basic z-transposition formula, where $\Phi = \{0, 6\}$ and $\Psi = \{0, 1, 3, 4\}$, or as a pair of all-interval tetrachords pumped by $p2$.

Pumping algorithms 3-4 take a different strategy: instead of pumping into the formula additional cyclic collections or remainder sets, algorithms 3-4 pump a symmetrical set (S) into the formula. The index of symmetry of S is either dependent on either x ($p3$), or on the index of symmetry of the remainder set ($p4$). Algorithm $p3$ is based on O'Rourke's, et al. (2008) pumping lemma; however, since for $p3$ the index of symmetry is dependent on the cycle and x, and not on pitch classes that have what they call 'isospectral vertices', $p3$ is applicable to more starting pairs than is

O'Rourke's pumping lemma. That is, with $p3$ the index of symmetry of S is not restricted to being around pitch classes that are present within the starting pair of sets.

Pumping algorithm 3:

Let $S \subseteq \mathbb{R}$ be a pitch-class set that is symmetrical at the index $d\phi$. Add $x + S$ to α and $-x + S$ to β.

$$\alpha \uplus x+S \quad \mathcal{Z} \quad \beta \uplus -x+S$$

Take, for example, the starting pair with Φ = {0, 6} and Ψ = {0, 1, 5}. To pump the pair with $p3$, we need to choose a set S that is symmetrical at either 0 or 6. Let us use the set S = {11, 11.5, 0, 0.5, 1}, and define x as x = 2. (Here I am choosing real numbers for S simply to show that S does not necessarily have to be integers.) Thus, according to algorithm $p3$:

$$\{0, 6\} \uplus 2 + \{0, 1, 5\} \uplus 2 + \{11, 11.5, 0, 0.5, 1\} \;\; \mathcal{Z} \;\; \{0, 6\} \uplus -2 + \{0, 1, 5\} \uplus -2 + \{11, 11.5, 0, 0.5, 1\}$$

$$\{0, 1, 1.5, 2, 2, 2.5, 3, 3, 6, 7\} \;\; \mathcal{Z} \;\; \{9, 9.5, 10, 10, 10.5, 11, 11, 0, 3, 6\}$$

Besides showing an instance of $p3$, this last example also shows that S (like Y) does not have to contain integer values. Though the resulting pair breaks beyond the mod12 pitch-class space and cannot be described using the traditional interval vector, it is z-related in a 'continuous harmonic space', as discussed by Callender 2007. In this case, the resulting pair could be described in mod24 space by doubling the values of each of the pitch classes. However, such a translation between spaces would not be possible when, for example, S = {π, –π}, which requires a continuous space for any given value of x.

Algorithm $p4$ is like $p3$, except that it uses a different index of symmetry for the set S. For $p4$, the index of symmetry is dependent on the index of symmetry of the remainder set Ψ; thus, $p4$ is applicable only to starting pairs that have a Ψ that is symmetrical.

Pumping algorithm 4:

> (Only for formulas that have symmetrical remainder sets.) Let i be the index of symmetry of Ψ. Let $S \subseteq \mathbb{R}$ be a pitch-class set that is symmetrical at the index $i + d\phi$. Add S to one side and $i–S$ to the other.

$$\alpha \uplus S \quad \mathcal{Z} \quad \beta \uplus i - S$$

For instance, let $\Phi = \{0, 6\}$ and $\Psi = \{0, 2, 4\}$. The index of symmetry of Ψ is $i = 4$, and thus S can be symmetrical at either indexes 4 or 10. Let S be a set that is symmetrical at $i = 4$, such as $S = \{2\}$. Thus, according to algorithm $p4$,

$$\{0, 6\} \uplus x + \{0, 2, 4\} \uplus \{2\} \;\; \mathcal{Z} \;\; \{0, 6\} \uplus -x + \{0, 2, 4\} \uplus 4 - \{2\}$$
$$\{0, 2, 6\} \uplus x + \{0, 2, 4\} \;\; \mathcal{Z} \;\; \{0, 2, 6\} \uplus -x + \{0, 2, 4\}$$

Since in this case $S = i - S = \{2\}$, the sets S and Φ can be combined, making the formula a sum of two sets instead of three.

Algorithm $p5$ is unlike the previous four algorithms, in that it begins with a formula where the two sides are equal (as opposed to z-related). Instead of turning a pair of z-related sets into a pair of larger cardinality z-related sets, $p5$ turns a single z-related set into a pair of larger cardinality z-related sets. Also, since algorithm $p5$ is dependent on the structure of the remainder set, it is more difficult to generalize than the other algorithms. For this reason, I offer first a preliminary version that incorporates the cyclic-sub z-transformation, which is non-specific to the details of

the remainder set, and then two rectified versions of the algorithm that are tailor-made for specific symmetrical remainder sets. First, let us take a look at the preliminary version of the algorithm.

Pumping algorithm 5 (preliminary version):

Begin with a formula where the two sides are equal.

$$\Phi \uplus x + \Psi \qquad = \qquad \Phi \uplus x + \Psi$$

Given the remainder set Ψ, let Ψ′ be a valid remainder set that literally includes Ψ as a subset. Add to both sides of the formula the pitch classes in $x + \Psi'$ that are not present in $x + \Psi$.

$$\Phi \uplus x + \Psi' \qquad = \qquad \Phi \uplus x + \Psi'$$

Apply the cyclic-sub z-transformation to one side of the formula.

$$\Phi \uplus x + \Psi' \qquad \mathcal{Z} \qquad \Phi \uplus x + \mathrm{CS}(\Psi')$$

The preliminary version of algorithm *p*5 is not actually a proper pumping algorithm, since the cyclic-sub could involve pitch classes in Ψ, in which case one of the two resulting z-related sets might not include the starting set as a subset. However, with certain remainder pairs the algorithm can be restated in a manner akin to algorithms 3 and 4 that guarantees that both resulting z-related sets will include the starting set. This is possible, however, for only a hand full of symmetrical remainder sets: I am aware of only two remainder sets in mod12 that can be pumped by algorithm *p*5—{0, 3} and {0, 2}. With these sets, the algorithm can be expressed as the union of the starting set and another symmetrical set *S* that is symmetrical at a particular index.

One side of the formula adds the symmetrical set *S*, while the other adds an inversion of *S*, invoking the cyclic-sub z-transformation. The two remainder sets ({0, 3} and {0, 2}) are used in two separate algorithms, *p*5a and *p*5b, which for convenience assume mod12.

Pumping algorithm 5a:

Let $\Phi = \{0, 6\}$ and $\Psi = \{0, 3\}$, and use a formula where the two sides are equal.

$$\Phi \uplus x + \Psi \qquad = \qquad \Phi \uplus x + \Psi$$

Let $i = 2x - 3$, and S be a set that is symmetrical at index i. Add S to one side of the formula, and $i + 6 - S$ to the other.

$$\Phi \uplus x + \Psi \uplus S \qquad \mathcal{Z} \qquad \Phi \uplus x + \Psi \uplus i + 6 - S$$

Pumping algorithm 5b:

Let $\Phi = \{0, 4, 8\}$ and $\Psi = \{0, 2\}$, and use a formula where the two sides are equal.

$$\Phi \uplus x + \Psi \qquad = \qquad \Phi \uplus x + \Psi$$

Let $i = 2x - 2$, and S be a set that is symmetrical at index i. Add S to one side of the formula, and $i + 4 - S$ to the other.

$$\Phi \uplus x + \Psi \uplus S \qquad = \qquad \Phi \uplus x + \Psi \uplus i + 4 - S$$

Let us take a look at an example of algorithm *p*5a, using the set {0, 1, 4, 6} (with $x = 1$) as the starting set. To pump the set, we choose a set that is symmetrical at index $i = 2(1) - 3 = 11$, such as the set {3, 8}. One side adds *S* while the other adds *S* inverted at index 6+11 = 5.

$$\{0, 1, 4, 6\} \uplus \{3, 8\} \;\; \mathcal{Z} \;\; \{0, 1, 4, 6\} \uplus 5 - \{3, 8\}$$
$$\{0, 1, 3, 4, 6, 8\} \;\; \mathcal{Z} \;\; \{0, 1, 2, 4, 6, 9\}$$

The algorithm in effect turns the starting set into a z-related pair that has a remainder set $\Psi \uplus S$, and that is related by the cyclic-sub z-transformation. While pitch-classes {1, 4} remain constant between the two sides (as well as the cyclic collection, {0, 6}, of course), the pitch classes from S and i+6–S invoke the cyclic-sub of two larger remainder sets (one on each side of the formula) that are of the same set class: {1, 3, 4, 8} and {1, 2, 4, 9} are both of set class [0237], and are related by cyclic substitution.

Though algorithm *p*5 can only be applied to only a couple z-relation formulas, the other four algorithms can be applied to a larger number of formulas. Though there very well could be more possible pumping algorithms, these five together cover a vast array of z-related pairs. One particular limitation is that these pumping algorithms can only begin with a formula in the form of Formulas 3.2 and 3.5. That is, the only z-related sets that can be pumped are those that have one cyclic collection. Nevertheless, despite this limitation, pumping offers much insight on the superset/subset structure of z-related sets.

To give a picture of the scope of these pumping algorithms, Table 3.3 presents all of the z-related pairs in mod12, showing their derivations as pumped all-interval tetrachords. As can be seen, pumping the all-interval tetrachords generates a majority of the z-related pairs in mod12—all except the three pentachordal pairs, four hexachordal pairs and one heptachordal pair.

Table 3.3: All of the z-related pairs in mod12 considered as pumped all-interval tetrachords.

Pentachords

Fourier partition	Set class	AIT union	Set class	AIT union	Pumping algorithm
[06] + [024]	[01356]	–	[01247]	–	n/a
[06] + [015]	[01457]	–	[01258]	–	n/a
[048] + [02]	[01348]	–	[03458]	–	n/a

Hexachords

Fourier partition	Set class	AIT union	Set class	AIT union	Pumping alg.	Notes
[06] + [06] + [03]	[012567]	α ∪ {11, 5}	[012378]	β ∪ {1, 7}	$p1$, $Y = \{-1\}$	
[06] + [0134]	[012456]	α ∪ {2, 5}	[012348]	β ∪ {10, 1}	$p2$, $Y = \{2\}$	
[06] + [0358]	[013578]	α ∪ {8, 11}	[012579]	β ∪ {4, 7}	$p2$, $Y = \{-4\}$	
[06] + [0135]	[012457]	α ∪ {11, 3}	[012358]	β ∪ {9, 1}	$p3$, $i = 0$, $S = \{10, 2\}$	
[06] + [0148]	[013478]	α ∪ {5, 9}	[012569]	β ∪ {3, 7}	$p3$, $i = 0$, $S = \{4, 8\}$	
[06] + [0124]	[012356]	α ∪ {3, 5}	[012347]	β ∪ {1, 3}	$p3$, $i = 6$, $S = \{2, 4\}$	
[06] + [0247]	[013568]	α ∪ {9, 11}	[012479]	β ∪ {7, 9}	$p3$, $i = 6$, $S = \{4, 8\}$	
[06] + [06] + [04]	[012568]	α ∪ {5, 10} β ∪ {7, 8}	[012478]	β ∪ {5, 10} α ∪ {7, 8}	$p4$, $S = \{5, 10\}$ $p4$, $S = \{7, 8\}$	
[06] + [06] + [02]	[012467]	α ∪ {2, 7} β ∪ {4, 5}	[012368]	β ∪ {1, 8} α ∪ {10, 11}	$p4$, $S = \{2, 7\}$ $p4$, $S = \{4, 5\}$	
[06] + [0125]	[013457]	β ∪ {3, 4}	[023458]	β ∪ {9, 10}	$p5$a, $S = \{3, 4\}$	
[06] + [0237]	[013468]	α ∪ {3, 8}	[012469]	α ∪ {2, 9}	$p5$a, $S = \{3, 8\}$	
[06] + [06] + [01]	[013467]	α ∪ {3, 7} β ∪ {3, 5}	[012369]	–	n/a	α ∪ 3+{0, 4} β ∪ 3+{0, 2}
[06] + [06] + [05]	[023679]	α ∪ {7, 9} β ∪ {5, 9}	[014679]	–	n/a	α ∪ –3–{0, 2} β ∪ –3–{0, 4}
[06] + [06] + [02]	[023568]	α ∪ {3, 10} β ∪ {8, 9}	[023469]	–	n/a	α ∪ 3–{0, 5} β ∪ –3–{0, 1}
[06] + [06] + [04]	[013569]	α ∪ {9, 10} β ∪ {3, 8}	[013479]	–	n/a	α ∪ –3+{0, 1} β ∪ 3+{0, 5}

Heptachords

Fourier partition	Set class	AIT union	Set class	AIT union	Pumping algorithm
[06] + [06] + [024]	[0123568]	α ∪ {3, 5, 10} β ∪ {9, 10, 11}	[0123479]	β ∪ {1, 3, 8} α ∪ {7, 8, 9}	$p3$, $i = 6$, $S = \{2, 4, 9\}$ $p3$, $i = 6$, $S = \{8, 9, 10\}$
[06] + [06] + [015]	[0124578]	α ∪ {3, 7, 11} β ∪ {3, 5, 7}	[0145679]	β ∪ {1, 5, 9} α ∪ {5, 7, 9}	$p3$, $i = 0$, $S = \{0, 4, 8\}$ $p3$, $i = 0$, $S = \{4, 6, 8\}$
[048] + [06] + [02]	[0134578]	β ∪ {4, 5, 8}	[0124569]	α ∪ {2, 5, 9}	no algorithm

Table 3.3, cont..

Octachords

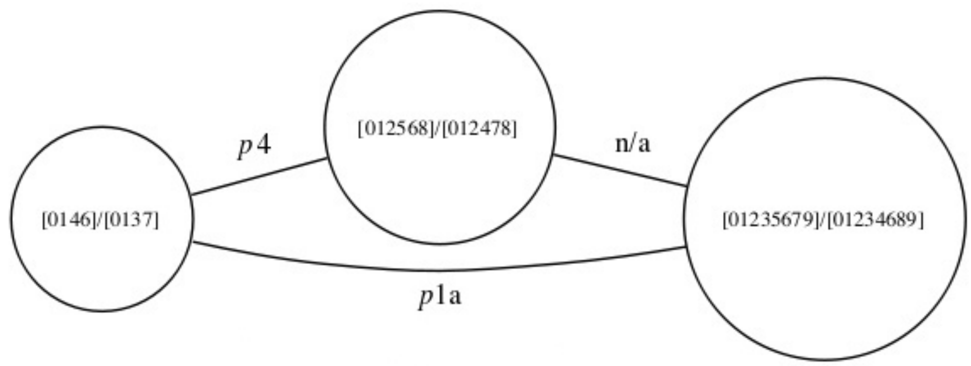
Fourier partition	Set class	AIT union	Set class	AIT union	Pumping algorithm
[06] + [06] + [06] + [03]	[01235679]	α ∪ {2, 5, 8, 11}	[01234689]	β ∪ {1, 4, 7, 10}	$p1$, Y = {–1, 2}, or $p2$, Y = {–1, 5}
		α ∪ {3, 5, 9, 11}		β ∪ {1, 3, 7, 9}	$p1$, Y = {–1, 3}, or $p3$, S = {2, 4, 8, 10}
		α ∪ {2, 5, 7, 10}		β ∪ {1, 5, 8, 10}	$p2$, Y = {–5, 2}
		α ∪ {7, 8, 10, 11}		β ∪ {4, 5, 7, 8}	$p2$, Y = {–5, –4}
		β ∪ {3, 4, 9, 10}		α ∪ {2, 3, 8, 9}	$p1$, Y = {2, 3}
		β ∪ {1, 4, 5, 8}		α ∪ {7, 10, 11, 2}	$p2$, Y = {–1, –5}
		β ∪ {5, 7, 8, 10}		α ∪ {5, 7, 8, 10}	$p4$, S = {5, 7, 8, 10}

Some of the z-related pairs, such as the heptachordal and octachordal pairs, can be generated in several different ways, by pumping the all-interval tetrachords with different algorithms. For instance, there are 8 different ways to pump the all-interval tetrachords (with algorithms *p*1 through *p*4) to make the octachordal pair, which is the complement of the all-interval tetrachords.

Pumping shows that there are cross-cardinality relationships between z-related pairs. If one z-related set includes a smaller z-related set as a subset, then in many cases the z-related partner will include the partner of the smaller z-related set. Z-related pairs beget more z-related pairs, either as subsets or as supersets. In this way, we can visualize a web of inter-cardinality relationships between z-related sets, either as a network or as families. Take, for instance, Example 3.17, which shows the pumping relationships between three z-related pairs: the all-interval tetrachords (left), their complements (right), and the all-trichord hexachord and its complement (center). The all-interval tetrachords generate both of the two larger z-related pairs. The all-trichord hexachord (ATH) and its complement, however, cannot be pumped to make the complemental pair of the all-interval tetrachords since it contains two cyclic-

collection subsets. Nevertheless, the ATH and its complement are subsets of the complement of the all-interval tetrachords, even though there is no pumping algorithm to describe their relationship.

Example 3.17: Example of set inclusion that involves pumping.



Networks like the one shown as Example 3.17 could be extended to include all of the z-related pairs in mod12. The complete super network (as we might call it) would generally divide into two parts—one with all of the z-related sets that have {0, 6} subsets, and those that have a {0, 4, 8} subset—however, there would even be subset/superset relationships, some being instances of pumping, between sets in the two different parts. Smaller networks, such as Ex. 3.17, are subsets of the complete network. Though the complete network would be interesting for it breadth, I will leave that for the reader. Besides, the smaller networks are more appropriate for music analysis, since it is rare (if even at all) that we find a piece of music that uses *all* of the z-related sets. Rather, in a given piece of music that incorporates z-related sets, it is usual for the composition to center around only a few select z-related pairs.

3.6 Reciprocal set unions

The algebraic methods presented thus far do not cover all of the z-related pairs in mod 12: there remain 4 hexachordal pairs (those with a set that has an [0369] subset). These 4 pairs, however, can be explained using another method that applies to sets that have more than one cyclic-collection subset (in the Fourier partition), and divide the modulus in half (that is, cardinality 6 in mod12). A remainder set for this type of z-related pair does not follow the same 4 criteria for Ψ given above, so it does not form z-related sets when paired with only one cyclic collection. Rather, z-related pairs arise from such remainder sets only when the remainder set is combined with two (or more, in some moduli) cyclic collections. Since the remainder sets are not valid Ψ, the intervals between the various subsets cannot be ascribed as a variable (such as x), and thus these pairs cannot be generalized in the same way that the other z-related pairs have been. For this reason, the discussion to follow is merely a description of the underlying structure of these z-related pairs (using an original solution), and does not provide generalized formulas involving the variable x.

First, let us consider the z-related hexachordal pairs in mod12 that have a Fourier partition with two cyclic-collection subsets (see Table 3.4). There are 7 such pairs, and they all are comprised of two instances of [06] and some dyad. Some of these pairs have already been covered by pumping, such as the pair with the [06]+[06]+[03] partition; however, all 7 pairs arise from a particular combinations of the subsets, what I call a *reciprocal set union*. In respect to each other, the sets of

each z-related pair have two subsets (one cyclic collection and the dyad remainder set) that are swapped in relation to a common cyclic collection.

Table 3.4: The 7 z-related hexachordal pairs in mod12 that have a Fourier partition with 2 cyclic-collection subsets.

Z-related pair		**Fourier partition**
[013467]	[012369]	[06] + [06] + [01]
[012467]	[012368]	[06] + [06] + [02]
[023568]	[023469]	[06] + [06] + [02]
[012567]	[012378]	[06] + [06] + [03]
[012568]	[012478]	[06] + [06] + [04]
[013569]	[013479]	[06] + [06] + [04]
[023679]	[014679]	[06] + [06] + [05]

To illustrate reciprocal set-unions, I use a graphic notation that represents each of the subsets as some shape, connected by lines representing intervals (see Example 3.18). Each set of a z-related pair is represented as two circles and a square, indicating the two cyclic collections (A and B) and the remainder set (C). The subsets are situated so that the subsets B and C lie at intervals *u* and *v* from the static cyclic-collection subset A (judging by the lowest pitch classes of the subsets in normal form). Both subset unions in Ex. 3.18 have the same subsets and intervals; however, in respect to the each other, the two graphs have the subsets B and C in opposite locations, with the intervals *u* and *v* switched between subsets A and B, and A and C. The corresponding z-transformation, then, is a function that moves by T_n (or T_nI) the subsets B and C to their respective positions in respect to the common cyclic collection.

Example 3.18: Basic model of reciprocal subset unions for z-related pair.

All 7 of the hexachordal pairs in Table 3.4 can be explained with this model, as can many other z-related pairs in other moduli. Some of the hexachordal pairs (in mod12) even have more than one solution using this model, being generated by two different values of *u* and *v*. Table 3.5 lists the 7 hexachordal pairs again, giving the values of *u* and *v* that produce these sets. As can be seen, two of the pairs can be generated by two sets of values for *u* and *v*.

Table 3.5: The 7 z-related hexachordal pairs in mod12 with 2 cyclic-collection subsets, and the values of *u* and *v* that generate them.

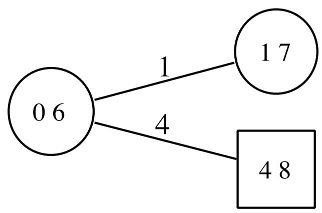
Z-related pair		**Fourier partition**	**Values of *u* and *v***	**Mapping under M5**
[013467]	[012369]	[06] + [06] + [01]	(1, 3)	[06] + [06] + [01]
[012467]	[012368]	[06] + [06] + [02]	(1, 2), (2, 5)	themselves
[023568]	[023469]	[06] + [06] + [02]	(2, 3)	themselves
[012567]	[012378]	[06] + [06] + [03]	(1, 5)	each other
[012568]	[012478]	[06] + [06] + [04]	(1, 4), (4, 5)	themselves
[013569]	[013479]	[06] + [06] + [04]	(3, 4)	themselves
[023679]	[014679]	[06] + [06] + [05]	(3, 5)	[06] + [06] + [05]

As an example of a reciprocal set union, let us consider the hexachordal pair with the all-trichord hexachord, the pair [012478] and [012568], which has two possible values for *u* and *v*, (1, 4) and (4, 5) (see Example 3.19).

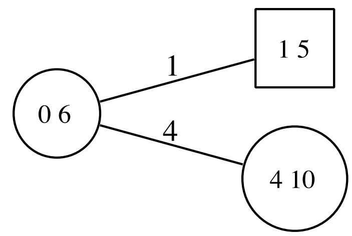
Example 3.19: Two possible reciprocal set unions of the subsets [06]+[06]+[04] that form sets of set classes [012478] and [012568].

a) reciprocal set union with intervals (1, 4)

[012478]

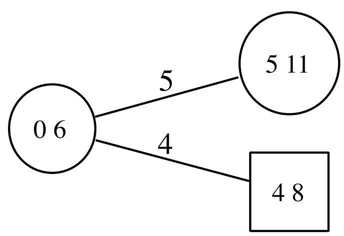


[012568]

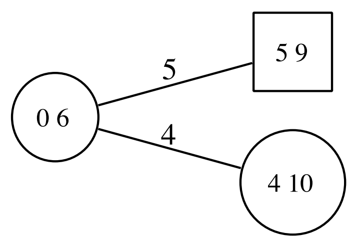


b) a) reciprocal set union with intervals (4, 5)

[012478]



[012568]



The intervals *u* and *v* are clearly dependent on the dyad subset; however, I am unable to offer an explanation as to why the particular intervals correspond to the dyad subsets of the various hexachordal pairs. On inspection, it appears that the interval i3 and the intervals i1 and i5 play an important role. When *u* or *v* is not 1, 3, or 5, then it is the interval of the dyad—e.g. the reciprocal set unions with dyad [02] both involve i2. Nevertheless, it is clear that the (*u*, *v*) values for the various reciprocal set unions follow a certain pattern, especially since they align with the M5-mappings (multiplication by 5) of the z-related pairs, which are shown in the right-hand column

of Table 3.5. For the single pair that uses values (1, 5) in the reciprocal set union, M5 maps the two sets in the pair onto each other—that is, M5 of an instance of [012567] yields a set that is of the set class [012378], and vice versa. The sets of the four pairs that have [02] or [04] in the Fourier partition each map onto themselves under M5, and the two pairs with [01] and [05] map onto each other under M5. The M5 mappings reflect exactly the pattern that underlies the possible (u, v) values, as both have a mirror-type structure circling around the z-related pair in the center of the table.

Though reciprocal set unions cannot be generalized (as far as I know), which would enable them to form an infinite number of z-related sets (like the algebraic formulas), they provide some insight into the structure of some of the z-related sets that cannot be described by the other postulated algebraic formulas. One weakness arises when reciprocal set unions are used for analysis: the reciprocal set-union requires a considerable amount of calculation when given only one set of the hexachordal pair. Suppose we were given the set {1, 4, 6, 7, 8, 10}, and were asked to find the z-related partner based on a reciprocal set union. We would need to figure out which of the two [06] subsets (of the [06]+[06]+[02] partition) is the static subset; but in order to do so we would need to calculate all the potential intervallic combinations that u and v could be for both the given set and the z-related partner. That is to say, without knowing the set class of the z-related partner, it is impossible to determine the values of u and v. Suppose, then, that we knew the values of u and v, which in this case are (2, 3). Now the task is manageable, but we still have to go through all the

potential intervallic combinations (though perhaps mentally) in order to figure out which [06] subset is the static subset. We would then see that:

$$\begin{aligned}\{1, 4, 6, 7, 8, 10\} &= \{4, 10\} \cup \{7, 1\} \cup \{6, 8\} \\ &= 4+(\{0, 6\} \cup 3+(\{0, 6\}) \cup 2+(\{0, 2\}))\end{aligned}$$

and the z-related partner that corresponds to the reciprocal set union would thus be:

$$\begin{aligned}4+(\{0, 6\} \cup 3+(\{0, 2\}) \cup 2+(\{0, 6\})) &= \{4, 10\} \cup \{3, 5\} \cup \{2, 8\} \\ &= \{4, 6, 7, 9, 10, 0\}\end{aligned}$$

Despite the limitations of the reciprocal set union, there are certainly musical situations that could be explained using the reciprocal set union. If nothing else, for several hexachordal pairs, the reciprocal set union is one of at least two possible explanations; for these pairs, then, the reciprocal set union can be considered as yet another means for explaining z-transformations in music analysis.

4. Applications

In the previous chapter I showed a number of algebraic equations that describe the relationships among z-related sets. The various forms of these equations represent what I call z-transformations, which are transformations that change a z-related set to its z-partner. The z-transformations are contextual transformations, in that they are dependent on the structure of the set onto which they are applied. Some of the z-transformations describe the relationships between z-related sets with one cyclic-collection subset (z-transposition, z-inversion and the cyclic-sub z-transformation), while another (the reciprocal set union) describes the relationship between sets of z-related pairs that evenly divide the modulus (such as the hexachords in mod12) and that have two cyclic-collection subsets (a total of 7 of 15 the z-related hexachordal pairs in mod12). In addition to the z-transformations I have also provided some pumping algorithms, which extend the equations, showing that inter-cardinality subset/superset relationships exist among the various z-related pairs. Together, the z-transformations and pumping explain many of the relationships between z-related sets, and thus are a powerful arsenal of analytical tools for tracking the various motions from one z-related chord to the next.

At this point, however, we have not yet seen how these tools can be used for music analysis. The role of this chapter is to demonstrate some real-world applications of the theory, and to share some of the musical insights that can be revealed. To begin, we will first revisit the z-transformations, looking at analytical examples to illustrate the most basic usages. Following that, we will investigate some

of the transformations developed by Guy Capuzzo in relation to the music of Elliott Carter. His transformations relate two instances of all-interval tetrachords that share either a [03] or [06] dyad, and are thus nearly identical to the z-transformations described here. After comparing Capuzzo's and my transformations, I move on to demonstrate pumping with the same example from the music of Carter. Next, I discuss a passage from Luciano Berio's *Sequenza IXb* to demonstrate an instance of a pair of sets with opposite-pointing Fourier coefficients that sum to make a superset that has FOURPROP(1), as well as some other z-transformation-based relationships. Then, I turn to Schoenberg's 3rd String Quartet, op. 30, to show how the present theory can inform the analysis of twelve-tone music. From the quartet, I show examples of row-form relationships based on both z-transformations and pumping. Lastly, I close the chapter with a short discussion on a heuristic approach for identifying z-related sets using the Fourier partition.

4.1 The z-transformations for the analysis of all-interval tetrachords

Though the present theory covers all of the z-related sets in mod12, I have chosen to start this chapter with an overview of the theory applied specifically to the all-interval tetrachords (AITs), the sets [0146] and [0137]. Not only are the relationships easier to comprehend (since after all there are only four notes in each collection), but the all-interval tetrachords also seem to appear together in music quite often. Though while it may be difficult to find examples of certain z-related pairs, there is no shortage of examples that include the all-interval tetrachords.

As mentioned in Chapter 1, the all-interval tetrachords have been central to much of the recent discourse on the music of Elliott Carter. Capuzzo (2000 and 2004) and Adrian Childs (2006), among others, have explored some of the properties of the all-interval tetrachords to trace how the two sets interact with each other in Carter's music, and ways in which one all-interval tetrachord is changed to the next. Following Morris 1987 and 1990, much of their work has focused on the subset-union properties of the all-interval tetrachords, and on the subsets shared between the tetrachords. They both observe, as did Morris, that the all-interval tetrachords both partition into [03] and [06] subsets, and that any non-overlapping combination of the two dyads yields one of the two all-interval tetrachords.

In one particular example in his 2004 article, Capuzzo introduced a series of transformations for all-interval tetrachords, which affect either the [03] or [06] subsets of one all-interval tetrachord (by transposition or inversion) to form another all-interval tetrachord that shares the other dyad subset. As we shall see, his transformations are in fact nearly identical to the z-transformations given here. However, in comparing his methodology and mine, we find that though the two corresponding analyses of the passage may differ on the surface only in orthography, my analysis based on the z-transformations more accurately reflects the underlying properties of the z-relation.

Before exploring this particular passage of music, however, let us take a moment to reevaluate the all-interval tetrachords under the group of z-transformations, and to look at another example from the music of Carter to get a

basic understanding of how the z-transformations can be used for the analysis of all-interval tetrachords. As we saw before, given a pair of z-related sets that share a partition that contains one cyclic collection, there are a certain number of other instances of the two z-related set classes that also have the same cyclic-collection pitch-class set as a subset. The total number of such instances (g) is dependent on the base cycle ([06] or [048], etc.), and on whether or not one (or both) of the sets of z-related pair is symmetrical. The complete collection of instances has a structure in the form of (where $0 < d <$ cardinality of Φ):

$$\Phi \uplus \begin{array}{l} x + \Psi \\ -x - \Psi \\ d\phi + (x + \Psi) \\ d\phi + (-x - \Psi) \end{array} \qquad \mathcal{Z} \qquad \Phi \uplus \begin{array}{l} -x + \Psi \\ x - \Psi \\ d\phi + (-x + \Psi) \\ d\phi + (x - \Psi) \end{array}$$

The z-related sets that involve $d\phi$ are related by transposition to the sets that do not have $d\phi$ (that is, the sets for which $d = 0$). To derive the z-transformations Z_T (z-transposition) and Z_I (z-inversion), let us reduce the collection to only four members (as shown in Chapter 3), conceptually grouping together the sets that are transpositionally related by $d\phi$. Accordingly, the reduced collection is simply:

$$\Phi \uplus \begin{array}{l} x + \Psi \\ -x - \Psi \end{array} \qquad \mathcal{Z} \qquad \Phi \uplus \begin{array}{l} -x + \Psi \\ x - \Psi \end{array}$$

Now, let us write out the collection specifically for the all-interval tetrachords, assigning Φ and Ψ to the pitch-class sets {0, 6} and {0, 3} and the variable x to some integer. (For the present discussion, I have chosen to let x be 1, though it could be any integer mod12.)

Example 4.1: The (reduced) collection of all-interval tetrachords that share a common [06] subset.

$$\{0, 6\} \cup \begin{matrix} 1 + \{0, 3\} \\ -1 - \{0, 3\} \end{matrix} \qquad \mathcal{Z} \qquad \{0, 6\} \cup \begin{matrix} -1 + \{0, 3\} \\ 1 - \{0, 3\} \end{matrix}$$

$$\begin{matrix} \{0, 1, 4, 6\} \\ \{6, 8, 11, 0\} \end{matrix} \qquad \mathcal{Z} \qquad \begin{matrix} \{11, 0, 2, 6\} \\ \{6, 10, 1, 0\} \end{matrix}$$

The two AITs on the left-hand side of the formulas in Example 4.1 are of the set class [0146] (related by inversion), and the ones on the right are of set class [0137]. For convenience, let us call the two sets on the left-hand side of the formula α and α* (where '*' is the bottom set in the formula), and the two on the right β and β*, as illustrated in Example 4.1.

Table 4.1 illustrates the group of transformations that relate the four AITs. For any one of the four AIT forms, the three non-identity transformations, Z_T, Z_I and I_C (contextual inversion) transform the form to one of the other three forms.

Table 4.1: The all-interval tetrachords under the z-transformations (X = identity).

	α	α*	β	β*
α	X	I_C	Z_T	Z_I
α*	I_C	X	Z_I	Z_T
β	Z_T	Z_I	X	I_C
β*	Z_I	Z_T	I_C	X

The table shows that, for example, Z_T takes α* to β*, and Z_I takes β to α* (reading the first set in the left column and the second in the top row). Specifically, Z_T is the contextual transformation that transposes the [03] subset of α and β* by T_2, or the [03] subset of α* and β by T_{-2}, Z_I is the contextual transformation that inverts the [03] subset about the pitch class that is closest (ic1) to one of the pitch classes of the [06] subset, and I_C is the contextual inversion that inverts the [03] subset around the conceptual "0" of the [06] subset (when x = 1, "0" is always the pitch of the [06] subset that is a half-step away from a pitch in the [03] subset).

To find compound z-transformations, we can consult the combination table shown as Table 4.2. As the table shows, performing Z_T twice, for example, results in the identity, and performing a Z_T followed by a Z_I is equivalent to I_C. Overall, the table shows that each of the z-transformations is its own inverse, and that the z-transformations are commutative (a*b = b*a).

Table 4.2: Combination table for the z-transformations.

	X	Z_T	Z_I	I_C
X	X	Z_T	Z_I	I_C
Z_T	Z_T	X	I_C	Z_I
Z_I	Z_I	I_C	X	Z_T
I_C	I_C	Z_I	Z_T	X

If we allow the four AIT sets (α, α*, β and β*) to be transposed, then the four sets serve to represent four generic AIT types, where each of the 48 AITs is one of the four types, and for each type there are 12 transpositionally related AIT representatives. Adding transposition allows us to extend the group of z-

transformations to relate any two instances of all-interval tetrachords, a feature that will be most useful in analysis. The new table is shown as Table 4.3.

Table 4.3: The z-transformations with transposition.

	α	α*	β	β*
α	T_n	T_nI_C	T_nZ_T	T_nZ_I
α*	T_nI_C	T_n	T_nZ_I	T_nZ_T
β	T_nZ_T	T_nZ_I	T_n	T_nI_C
β*	T_nZ_I	T_nZ_T	T_nI_C	T_n

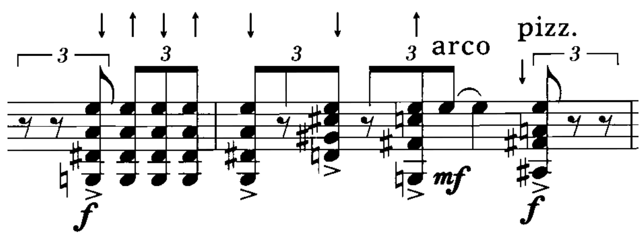
One effect of adding T_n to the z-transformations is that, like T_n itself, the z-transformations are no longer self-inverses—that is, applying T_nZ_T twice yields T_0 only when *n* is 0 or 6; otherwise, it yields T_{2n}. The new z-transformations, however, still form a commutative (a*b = b*a) group. I would show the combination table for the new group of transformations, using T_1 through T_{12}, but such a table is too large to print here, being a table with 48 rows and columns. Nevertheless, for now, we can just remember that the two inverses of the z-transformations T_nZ_T and T_nZ_I are $T_{-n}Z_T$ and $T_{-n}Z_I$, respectively, and the inverse of T_nI_C is $T_{-n}I_C$.

Before moving on to music analysis, I would also like to mention the other z-transformation that applies to z-related sets with one cyclic-collection subsets, the cyclic-sub z-transformation (Z_{CS}). As was noted in Chapter 3, Z_{CS} is always equivalent to one or more of the other z-transformations in the group. In the case of the all-interval tetrachords, there are two possible Z_{CS} transformations, one that transposes by T_6 the conceptual “0” of the [03] dyad, and one that transposes by T_6 the “3” of the [03] dyad—that is, $Z_{CS\{0\}}$ and $Z_{CS\{3\}}$. The transformation $Z_{CS\{0\}}$ is

equivalent to T_6Z_I and holds a [026] subset in common between the two all-interval tetrachords. $Z_{CS\{3\}}$ is equivalent to T_0Z_I and holds a [016] subset in common.

Let us now turn to consider a passage containing all-interval tetrachords—a passage from Carter's *Statement* (1999), from *4 Lauds* for violin (see Example 4.2)—to see the z-transformations used in analysis.

Example 4.2: Carter, *Statement*, mm. 68-69, four pizzicati chords that are all-interval tetrachords.



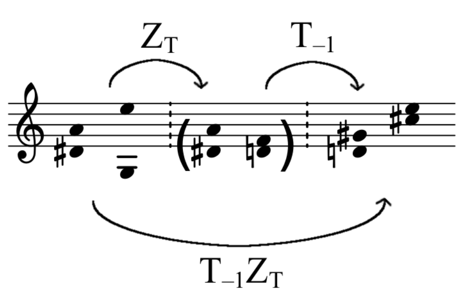
The passage consists of four pizzicati chords that are all-interval tetrachords, and that all share the open string E. To begin our analysis, let us first determine the AIT type for each chord. Though we could do this by considering the set class of each chord and determining whether or not it is inverted in relation to its prime form, the task can simply be accomplished by looking at how the [03] subset lines up with its corresponding [06] subset. Doing so, we find that among the four chords are all four types of AIT:

{D♯, E, G, A} – α
{C♯, D, E, G♯} – β
{C, E, F♯, G} – β^*
{E, F♯, A, A♯} – α^*

Having identified the AIT types for the four chords, we can already determine quite easily the transformation by which any two chords are related simply by consulting the table of transformations (Table 4.3). Accordingly, we can see that T_nZ_T (here T_n being some undetermined transposition) transforms the first chord to the second, T_nI_C the second chord to the third, and T_nZ_T the third chord to the fourth.

Like the usual set operators, when the transformations T_n and Z_T are written together, they are to be read from right to left. That is, the transformation T_nZ_T is a two-part process that begins with a Z_T—the [03] subset is transposed by either T_2 or T_{-2} depending onto which AIT type the Z_T is applied—and continues with a transposition by *n*. For example, let us consider the transformation between the first two chords in Example 4.2, which is shown in Example 4.3 as two steps: Z_T is applied first, which transposes the [03] subset by T_{-2}, and then all four pitch-classes are transposed by T_{-1}. As we can see, then, the complete transformation that takes the first chord to the (z-related) second chord, $T_{-1}Z_T$, is an amalgam of two steps, a z-transposition and a transposition.

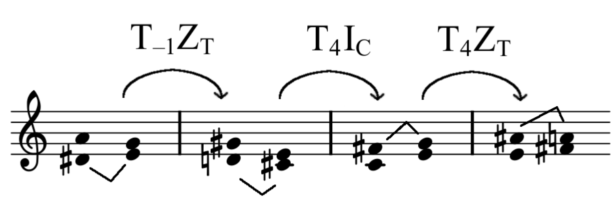
Example 4.3: The transformation of chords 1 and 2 from Carter's *Statement*, mm. 68-69, shown in two steps.



Just to be clear, however, it is also possible to switch the order of the two transformations (such as $Z_T T_n$), and perform the transposition first, and then the z-transposition; the result is always the same for Z_T, Z_I, as well as Z_{CS}. However, since I have found this way to be more difficult to visualize, I have chosen to always place the transposition last. Regardless of their order, we can also describe the transformation $T_{-1}Z_T$ as the transformation that when applied to α or β* transposes the [06] subset by T_{-1} and the [03] subset by T_{-3}—preserving the characteristic difference of –2 between the 'original position' and the shifted [03] subset—or that when applied to α* or β transposes the [06] subset by T_{-1} and the [03] subset by T_1. That is, T_nZ_T is some combination of two transposition levels, one applied to the [06] subset and the other to the [03] subset.

Using this methodology, let us turn back and work out the transformations between each of the four chords from the passage. Example 4.4 shows the chords with the registers of the pitch classes normalized to show more clearly the [06]+[03] structure of each chord. Small angle brackets show the location of the half step (i.e. *x*) within each collection.

Example 4.4: The z-transformations between the four chords.



Again, each transformation is to be visualized as a z-transformation, followed by the transposition of the whole set. In this example, we can also see that since the z-

transformations only affect the [03] subsets, the T_n levels are reflected in the transposition levels of the [06] subset: the tritone moves down a half step, then up a major third, and then up another major third. Knowing the three transformations between each chord, we can also easily determine the relationship between non-adjacent chords by simply adding together z-transformations according to the combination table, Table 4.2. For example, the transformation that takes the first chord onto the third is $T_{-1}Z_T(T_4I_C)$, which is equivalent to T_3Z_I. Similarly, the transformation that takes the first chord to the last is $T_4Z_T(T_3Z_I)$, which is equivalent to T_7I_C.

4.2 Capuzzo's transformations

In his 2004 article, Guy Capuzzo developed a series of transformations that show the relationships between various all-interval tetrachords. The analyses in which these transformations appear adopt a Lewinian transformational perspective, aiming to show the motions from one all-interval tetrachord to another. Like my z-transformations, his transformations are based on subset unions of the [03]+[06] dyad subsets of the all-interval tetrachords. Ultimately, as we shall see, our transformations are nearly identical. The goal of this section is to give an overview of his transformations, and to compare them to my z-transformations. As I show, though Capuzzo's transformations accurately describe certain relationships between all-interval tetrachords, they are formulated is such a way that they can only be applied to the all-interval tetrachords. My z-transformations clarify and extend Capuzzo's transformations, then, since they reflect the underlying principles behind the

relationships. Unlike Capuzzo's transformations, my transformations are generalized and can be applied to other z-related pairs besides just the all-interval tetrachords.

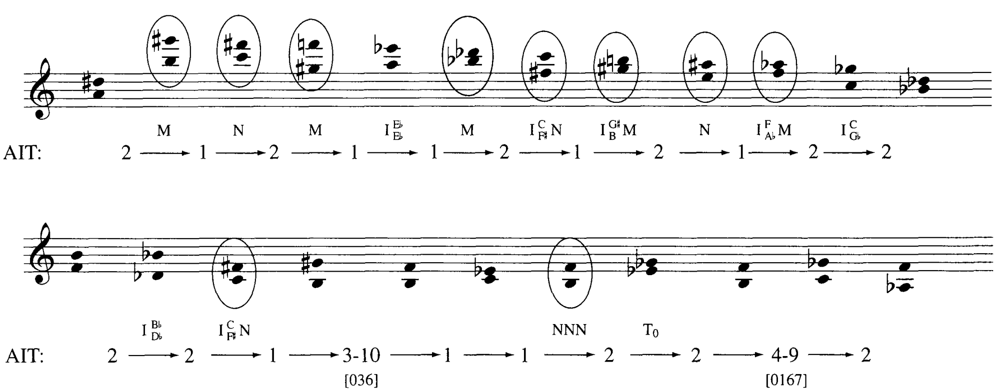
Let us begin by looking at the music passage for which Capuzzo's transformations were originally designed: a passage from *Fantasy—Remembering Roger* (1999), from *4 Lauds* for violin (see Example 4.5). The passage is an example of one compositional technique that Carter often uses, which is to chain together all-interval tetrachords, overlapping pitch classes from one all-interval tetrachord to the next. Though it is possible to chain all-interval tetrachords that share 1, 2 or 3 common pitch classes, this passage from *Fantasy* has a long chain based exclusively on dyad overlap involving the [03] and [06] subsets, which, as Carter himself pointed out in *Harmony Book* (2002), is the only shared dyadic partition between the two all-interval tetrachords.

Example 4.5: Carter, *Fantasy—Remembering Roger,* mm. 55-63, with [03] and [06] dyads marked.

In mm. 56-61, all of the pairs of adjacent dyads form all-interval tetrachords, except for one case where the [03] and [06] dyads share one pitch class (m. 60, marked by the circle), and one case where there are two adjacent [06] dyads (m. 61).

In his analysis of this passage, Guy Capuzzo used a series of transformations that connect the adjacent all-interval tetrachords in the chain (see Example 4.6). He described all of the motions from one all-interval tetrachord onto another using only two basic types of transformations, M/N and inversion. The transformation M transposes the [06] dyad by T_{-3}, and N transposes the [03] dyad by T_{-3}—that is, the M and N transformations both transpose by T_{-3}, but they transpose different dyads.

Example 4.6: Capuzzo's analysis (2004, p. 16) of the passage from Carter's *Fantasy*.



In the example, each all-interval tetrachord is marked as '1' or '2', indicating either AIT_1 ([0146]) or AIT_2 ([0137]). The circled dyads indicate which dyads act as pivots between the two AIT set classes. Between each of the adjacent AITs, Capuzzo indicates the transformation that takes one all-interval tetrachord onto the next, using only four transformations, M, N, IM or IN—or some multiple thereof (such as NNN). The inversion I is limited to either the inversion that maps the two pitch classes of the

shared subset onto each other, or in some cases in which [06] is shared, the inversion that maps each pitch class of the [06] subset onto itself.

The M and N transformations are indeed universally true for any all-interval tetrachord. To put it more broadly than does Capuzzo, a transposition by $T_{\pm 3}$ of either dyad of an AIT will always yield an instance of the other AIT set class. In this regard they are proper z-transformations. The M and N transformations, however, do not necessarily hold true for any z-related pair. Ultimately, the reason that these two transformations exist at all for the all-interval tetrachords is that they are equivalent to the two cyclic-sub z-transformations. That is, given the cycle [06], then the following is true:

$$
\begin{aligned}
3 + \{0, 3\} &= CS_{\{0\}}(\{0, 3\}), \text{ and} \\
-3 + \{0, 3\} &= CS_{\{3\}}(\{0, 3\})
\end{aligned}
$$

If the levels of transposition of the M and N transformations were defined so that they reflected the cyclic-sub z-transformations, then the M and N transformations would in fact be true for other z-related sets in mod12—specifically those that have one cyclic-collection subset and a remainder set that is inversionally symmetrical. Take, for instance, the z-related pair based on the subsets [06]+[0358]: the remainder set {0, 3, 5, 8} maps onto another [0358] under $CS_{\{0, 5\}}$, mapping onto the set {3, 6, 8, 11}, or under $CS_{\{3, 8\}}$, mapping onto the set {9, 0, 2, 5}. These two sets produced by CS are indeed related by $T_{\pm 3}$ to the original remainder set. Thus, like the all-interval tetrachords, a set of this z-related pair always maps onto a z-related partner by transposing either the remainder set or the cyclic collection by $T_{\pm 3}$. On the other hand,

for the z-related pair based on the subsets [06]+[024], transposing the remainder set by $T_{\pm 3}$ does not create the z-related partner. Instead, transposing by T_2 (or T_{-2}) finds the z-related partner, since $T_{\pm 2}$ corresponds to the cyclic substitutions of [024]—$CS_{\{0\}}\{0, 2, 4\} = T_2\{0, 2, 4\}$ and $CS_{\{4\}}\{0, 2, 4\} = T_{-2}\{0, 2, 4\}$.[1]

Before we reexamine the passage from *Fantasy* using my z-transformations, let us also consider a reformulation of the z-transformations that instead holds the remainder set in common and moves the cyclic collection. Such a formulation is an optional adaptation, but it is helpful for this particular passage since there are many cases in which the [03] dyad is held in common between two successive AITs. The group that holds the [03] is expressed as that shown in Example 4.7.

Example 4.7: Alternate formula that holds [03] subsets in common.

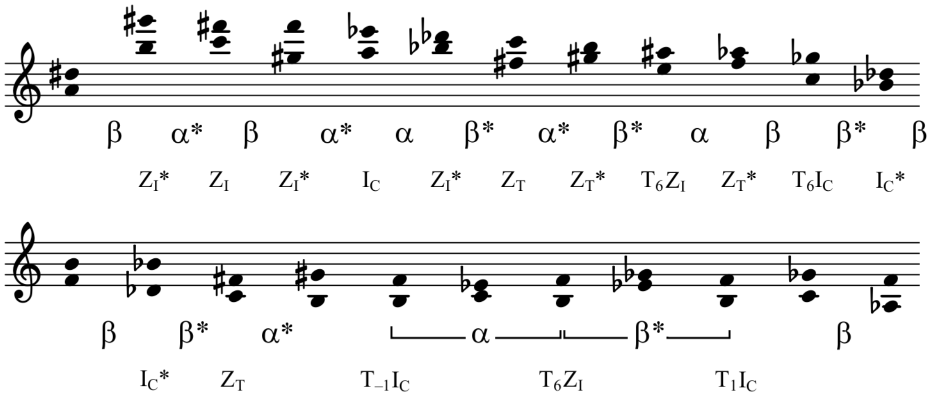
$$\{0, 3\} \cup \begin{matrix} -1 + \{0, 6\} \\ 4 - \{0, 6\} \end{matrix} \qquad \mathcal{Z} \qquad \{0, 3\} \cup \begin{matrix} 1 + \{0, 6\} \\ 2 - \{0, 6\} \end{matrix}$$

Still, the two sets on the left-hand side are inversionally related instances of set class [0146], and the two on the right are instances of [0137]. Z-transposition is the same as before (as $\pm x$), except that $x = -1$ now forms the set α and $x = 1$ forms β; the contextual inversion (the relationship between two sets on one side), however, now flips the set about the axis that maps the two pitch classes of the [03] dyad onto each other. To differentiate the z-transformations that hold [03] in common from the z-transformations that hold the [06] in common, I will write the former with an added asterisk '*'—such as Z_T* or Z_I*.

[1] See Chapter 3.3 for a discussion on the limitations of the cyclic-sub z-transformation on this particular z-related pair with the [024] remainder set.

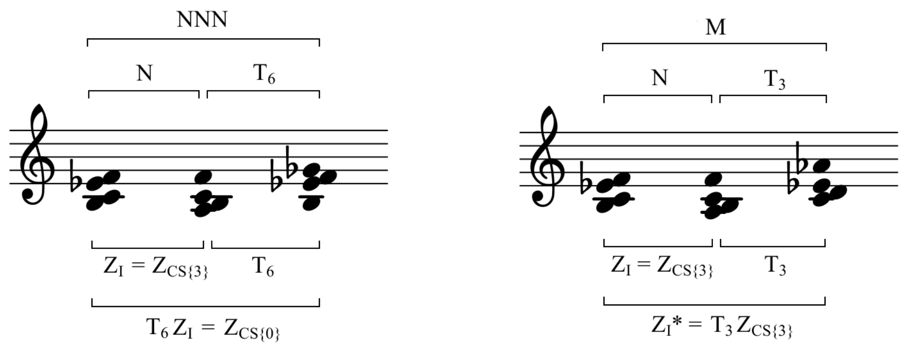
Using these two versions of z-transformations (where one holds [06] in common and the other [03]), let us now revisit the passage from *Fantasy*. Example 4.8 shows an analysis of the passage using z-transformations.

Example 4.8: Carter, *Fantasy*, mm. 55-63, annotated with the 4-types of all-interval tetrachord and the z-transformations between overlapping or adjacent AITs.



In Example 4.8, each of the all-interval tetrachords is labeled as one of the four types (α, α*, β and β*), and the relationships between adjacent all-interval tetrachords are labeled as z-transformations. If we compare Capuzzo's and my analyses of the passage (Examples 4.6 and 4.8), we can see that the two analyses line up, essentially showing the same relationships. Capuzzo's M is my Z_I*, his N is my Z_I, his IM is my Z_T*, and his IN is my Z_T. Example 4.9 illustrates the relationship between the Capuzzo's transformations and the z-transformations, showing that the there is a direct correspondence between our transformations.

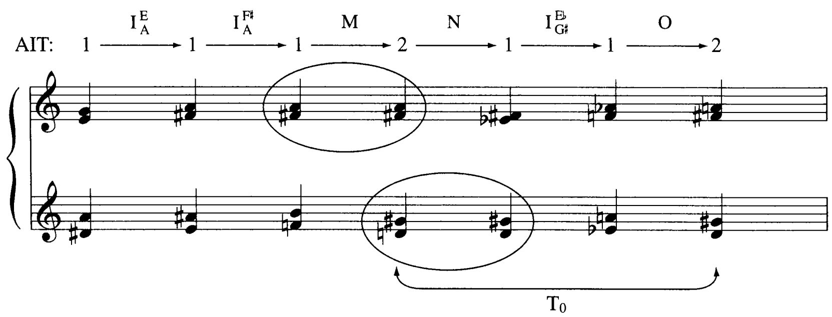
Example 4.9: A comparison of Capuzzo's transformations (above) and the z-transformations (below).



For this particular passage of music, Capuzzo's transformations are perhaps better suited to describe the relationships among the all-interval tetrachords, since the passage indeed includes many instances where subsequent [03] or [06] dyads are transposed by T_{-3}. Also, with this passage, Capuzzo's transformation cover all the relationships without having to invoke transposition; that is, there is no T_nN, for example, in his analysis of the passage. In these regards, Capuzzo should be credited for having devised a schema that not only captures the relationships between all-interval tetrachords, but that does so quite elegantly.

By being based solely on the N and M transformations, however, Capuzzo's transformations do not cover what I describe as z-transposition. Using his notation, the Z_T (and Z_T*) transformations are either IM or IN. The consequent group structure that he outlines is thus based entirely on the cyclic-sub z-transformation. Later in his article he does introduce another transformation that comes close to z-transposition, in an analysis of a passage from Carter's *Scrivo in vento* (1991). The transformation, O, transposes the [03] dyad by T_1 and [06] dyad by T_{-1} (see Example 4.10).

Example 4.10: Analysis of passage from Carter's *Scrivo in vento* (Capuzzo 2004, 18).



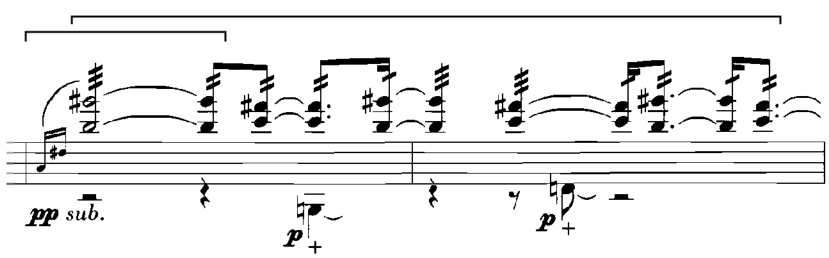
Expressed in terms of the z-transformations, the transformation O in Example 4.10 would be $T_{-1}Z_T$. However, instead of keeping one dyad in common, as he did before, Capuzzo defined the transformation O as transposition of both dyads. If he had defined O as the transposition of only one dyad, like Z_T, he might have seen that the 6th and 7th transformations in his *Fantasy* analysis (Example 4.6) are in fact instances of O.

In short, the transformations that Capuzzo outlined in his 2004 article come very near to the z-transformations described here. However, his transformations are formulated in such a way that they are conceptually limited to the all-interval tetrachords, and do not sufficiently describe fundamental relationships held by other z-related sets in mod12. The present theory thus expands Capuzzo's work, showing that the relationships that he describes are based on profound aspects shared by many z-related sets.

4.3 An instance of pumping in Carter's *Fantasy*

The passage from Carter's *Fantasy*, just discussed, also includes a particularly clear instance of pumping, which appears in the first two measures (see Example 4.11). Underneath the first two all-interval tetrachords in the passage (in mm. 56-57) there are two open-string pizzicato notes (G and D). Though the first AIT does not appear directly over the two pizzicati pitches, there is strong reason to believe, as I argue below, that the two pizzicati pitches group with both AITs.

Example 4.11: Carter, *Fantasy*, mm. 56-57, with two AITs bracketed.



The two bracketed all-interval tetrachords in Example 4.11 are z-related: the first is of the set class [0137] (β), and the second is of the set class [0146] (α*). When the [05] pizzicato dyad is added to each of the AITs, the two supersets remain z-related; that is, the [05] pizzicato dyad pumps the AITs to make a new z-related hexachordal pair. The pair of hexachords are created by applying *p*4 algorithm to the all-interval tetrachords, using a chosen set *S* of the set class [05]. To see this, let us first look at the generic algorithm that pumps the AITs to the a hexachordal pair that is of the same set classes as the hexachordal pair from the Carter example.

$$\begin{aligned}
\text{Let}\quad \alpha &= \{0, 6\} \cup 1 + \{0, 3\} \\
&= \{0, 1, 4, 6\}, \\
\beta &= \{0, 6\} \cup -1 + \{0, 3\} \\
&= \{11, 0, 2, 6\}, \\
i &= 3 && \text{(index of inversional symmetry of } \{0, 3\}\text{)}, \\
\text{and}\quad S &= \{5, 10\} && \text{(set that is inversionally symmetrical at index } i\text{)}.
\end{aligned}$$

Then, the following pair is z-related:

$$\begin{aligned}
\alpha \cup \{5, 10\} \;&\mathcal{Z}\; \beta \cup 3 - \{5, 10\} \\
&\mathcal{Z}\; \beta \cup \{5, 10\}
\end{aligned}$$

The above formula generates an instance of each of the two z-related hexachordal set classes, [012478] (6-z17, the all-trichord hexachord) and [012568] (6-z43) that both share the {5, 10} and {0, 6} pitch-class sets as subsets. Since the two sets share two dyad subsets, we can rewrite the formula as the union of only two subsets, a [0157] and a [03]:

$$\{5, 6, 10, 0\} \cup \{1, 4\} \;\mathcal{Z}\; \{5, 6, 10, 0\} \cup \{11, 2\}$$

In the passage from *Fantasy*, however, there is an additional twist. Instead of overlapping a [0157] subset, as do the two sets derived from the pumping algorithm, the two pumped hexachords in the example overlap the [03] and [05] dyads, and thus a [0147] subset. From the perspective of the pumping algorithm, this is achieved by inverting one side of the formula so that the [03] is held in common instead of the [06]. That is,

$$\begin{aligned}
0 - (\{5, 6, 10, 0\} \cup \{1, 4\}) \;&\mathcal{Z}\; 9 + (\{5, 6, 10, 0\} \cup \{11, 2\}) \\
\{7, 8, 11, 2\} \cup \{0, 6\} \;&\mathcal{Z}\; \{7, 8, 11, 2\} \cup \{3, 9\} \\
\{\text{F}\sharp, \text{G}, \text{G}\sharp, \text{B}, \text{C}, \text{D}\} \;&\mathcal{Z}\; \{\text{G}, \text{G}\sharp, \text{A}, \text{B}, \text{D}, \text{D}\sharp\}
\end{aligned}$$

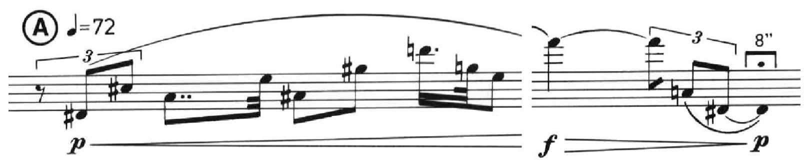
This last formulation gives us the two pumped AITs that appear in the passage. The hexachord that consists of the first AIT in the m. 56 and the G-D dyad is on the right-hand side of the formula, and the hexachord with the second AIT is on the left.

In this example, what is particularly interesting is the placement of the G-D dyad in relation to the two AITs, which supports a reading that suggests that the two AITs are indeed pumped. The union of the first AIT and the G-D dyad make an instance of the all-trichord hexachord, [012478] (6-z17). In Carter's music it is quite common to find all-trichord hexachords (ATHs) that have embedded all-interval tetrachords, and thus it is reasonable (and even perhaps correct) to assume that the G-D dyad originates compositionally as an extension of the first AIT to complete an ATH. However, the G-D dyad finally appears only once the upper AIT has already shifted, which forces us to group the dyad with the new AIT instead. Because of this, I argue, we can reasonably say that one of the functions of the [05] dyad is to act as a bridge between the two overlapping AITs at the beginning of the chain, pumping the two Z_I*-related AITs to a pair of z-related hexachords that includes the ATH.

4.4 Berio *Sequenza IXb*: sets with opposite pointing Fourier coefficients, and more examples of z-transformations

Like the Carter example above, one passage from *Sequenza IXb* for solo saxophone (1980) by Luciano Berio includes instances of overlapping all-interval tetrachords (see Example 4.12). In this passage, we also find an instance of sets with opposite-pointing Fourier coefficients.

Example 4.12: Luciano Berio, *Sequenza IXb* for solo E♭ saxophone, rehearsal A.



In the piece, the phrase (shown as Example 4.12) is repeated four times. Each of the repeated iterations of the phrase begins on the second pitch of the previous iteration, forming a gradual shortening process.

Let us first consider the first eight pitches of the first iteration of the phrase (in Ex. 4.12). The first four pitches of the phrase are an instance of [0137] (the following are written pitches, and not sounding pitches):

{D♯, C♯, A, E} => {9, 1, 3, 4} [0137]

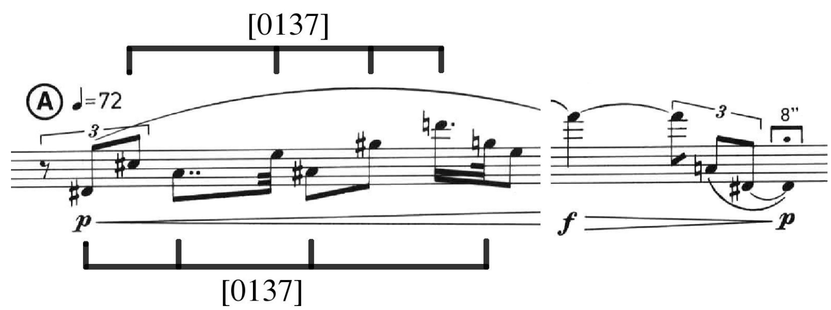
Following these first four pitches are another four that also form the set class [0137]:

{A♯, G♯, D, G} => {7, 8, 10, 2} [0137]

Not only do we find back-to-back [0137]s in these first eight pitches, but within these eight pitches there is also another pair of [0137]s, separated by register. The first eight pitches divide into two voices, an upper and a lower voice that alternate every other note (except that the last pair of pitches). Both the upper and voices are of set class [0137] (see Example 4.13). The first eight pitches of the phrase thus parse into two [0137]s in two different ways: either as pitches 1-4 and 5-8, or as upper and lower voices.

pitches 1-4 and 5-8	lower and upper voices
{D♯, C♯, A, E}, {A♯, G♯, D, G}	{D♯, A, A♯, G}, {C♯, E, G♯, D}

Example 4.13: Overlapping [0137]s at beginning of phrase.



At first glance, the partition scheme used in the passage may appear as a marvel of combinatorics; the Fourier transform, however, provides insight into how such a structure is possible. We need only to begin by considering the first eight pitches, which are of the set class [01236789] (Forte 8-9)—a set that has FOURPROP(1), consisting of four [06] subsets. Since the superset has FOURPROP(1), it follows that any 2-part partition will result in two subsets with $\mathbb{F}_1$ coefficients of the same magnitude and that point in opposite directions, even if the two sets are not of the same cardinality. Statistically speaking, then, a majority (if not all) of the 2-part equal-cardinality partitions (basically splitting the superset in half) are going to involve sets that are of same set class or that are z-related. With this particular case, the superset (Forte 8-9) has 17 different partitions with 2 equal-cardinality subsets, 12 of the same set class, and 5 that are of different set classes (see Table 4.4).

All 17 pairs of sets in Table 4.4 are two sets that have $\mathbb{F}_1$ coefficients with equal magnitudes and that point in opposite directions. For the set [01236789] there happens to be no 2-part partitions with z-related subsets. However, included in the 12 2-part partitions with subsets of the same set class are two partitions that have two

instances of either one or the other all-interval tetrachords. In total, any given instance of [01236789] has four unique 2-part partitions where both subsets in the partition are all-interval tetrachords (see Table 4.5). Again, like all 2-part partitions of supersets with FOURPROP(1), the two subsets of each partition have equal-magnitude $\mathbb{F}_1$ coefficients that point in opposite directions.

Table 4.4: The 2-part equal-cardinality partitions of the set class [01236789] (Forte 8-9).

a) partitions with subsets of the same set class

[0123]+[0123]	[0125]+[0125]	[0145]+[0145]
[0126]+[0126]	[0146]+[0146]	[0137]+[0137]
[0237]+[0237]	[0157]+[0157]	[0257]+[0257]
[0167]+[0167]	[0158]+[0158]	[0268]+[0268]

b) partitions with subsets of different set class

[0126]+[0236]	[0136]+[0127]	[0156]+[0147]
[0157]+[0258]	[0167]+[0369]	

Table 4.5: Partitions of {1, 2, 3, 4, 7, 8, 9, 10} (the first eight pitches of the Berio passage) with all-interval tetrachords.

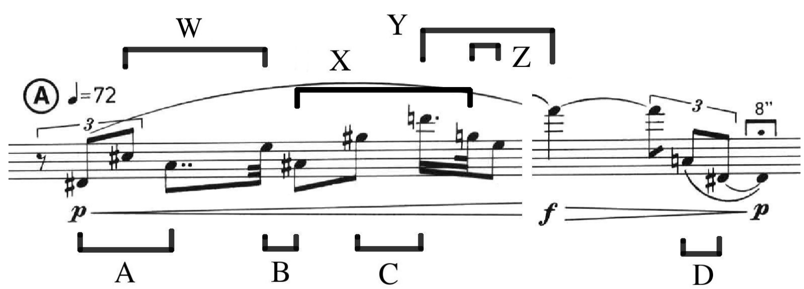
[0137] + [0137]	[0146] + [0146]
{1, 2, 4, 8} {3, 7, 9, 10}	{2, 4, 7, 8} {9, 10, 1, 3}
{9, 1, 3, 4} {7, 8, 10, 2}	{3, 4, 7, 9} {8, 10, 1, 2}

Though this example clearly demonstrates aspects of 2-part partitions of sets with FOURPROP(1), the example only involves the set class [0137], and thus does not specifically address the z-relation per se. However, since the principles apply to z-

related sets as well, I will now put aside the question of sets with opposite-pointing Fourier coefficients, to continue instead with some more examples of the z-transformations. Let us turn back to the passage in *Sequenza IXb*, and reexamine it with the aim of identifying some of the relationships between instances of the all-interval tetrachords.

After the first 8-pitches, there are a couple of particularly notable instances of [0146]. These two instances relate quite closely to other adjacent [0137]s, in that the relationships can be expressed directly by the z-transformations without having to use transposition—that is, T_n is not needed. To facilitate the identification of the various all-interval tetrachords, Example 4.14 marks some of the notable instances of [06] and [03], marking instances of [03] above the staff, and the instances of [06] below.

Example 4.14: Some notable [03] and [06] subsets in the phrase (*Sequenza IXb*) with [03] subsets marked above, and the [06]s below.



As it has already recognized, the all-interval tetrachords are a complement-union pair, in that any non-overlapping union of [03] and [06] yields either one or the other all-interval tetrachords. Naturally, the same follows for Example 4.14: all pairings of W, X, Y or Z with A, B, C or D result in one of the all-interval tetrachords, provided that the two subsets do not share any pitch classes. Table 4.6

shows some significant pairings in Example 4.14, omitting the ones where [03] and [06] cannot be reasonably grouped perceptually (being too far apart), and where the [03] and [06] dyads share a pitch class.

Table 4.6: Some combinations of [03] and [06] dyads that make all-interval tetrachords in the passage from Berio, *Sequenza IXb*.

	W	**X**	**Y**	**Z**
A	{A, C♯, D♯, E} [0137]	-	-	-
B	-	-	-	-
C	{C♯, D, E, G♯} [0137]	{G, G♯, A♯, D} [0137]	-	{D, E, G, G♯} [0146]
D	-	-	{D, D♯, F, A} [0137]	{D♯, E, G, A} [0146]

Of the all-interval tetrachords in Table 4.6, let us look carefully at two particular instances and interpret them using z-transformations. First, let us begin with the z-related pair DY and DZ. The two collections share the final [06] of the phrase (D♯-A), differing only by the [03] subset: DY has a D-F dyad, and DZ an E-G dyad. In respect to the D♯-A tritone, the transpositional relationship between the two [03]s—where one dyad is situated one semitone above the D♯ (the E-G dyad) and the other is one semitone below the D♯ (the D-F dyad)—directly reflects z-transposition. Assuming for a moment that D♯ is 0, then the relationship is as follows:

$$\{D\sharp, A\} \cup \{D\natural, F\} \;\; \mathcal{Z} \;\; \{D\sharp, A\} \cup \{E, G\}$$
$$\{0, 6\} \cup -1+\{0, 3\} \;\; \mathcal{Z} \;\; \{0, 6\} \cup 1+\{0, 3\}$$

With these two particular AITs, the Z_T relationship between them is emphasized on the surface by the registral difference between the [06] dyad and the two [03] dyads, which like with the first eight pitches of the passage suggests an upper and lower part.

Next, let us look at the z-related pair CX and CZ. Like the pair just discussed, these two sets share the [06] subset (C), and have a different [03] subset (X and Z). However, in this case the two all-interval tetrachords have three pitches in common (D, D♯ and G) since X and Z share a pitch class. The two pitch classes that are not shared by the sets (the pitch classes A♯ and E) are a tritone apart. In this regard, these two all-interval tetrachords clearly exemplify the cyclic-sub z-transformation: the two sets are related by a partial transposition in which only one pitch class is transposed by T_6. Using the notation presented in Chapter 3, then, the relationship between the two sets CX and CZ can be expressed as an equation:

$$\{G\sharp, D\} \cup \{G, A\sharp\} \;\mathcal{Z}\; \{G\sharp, D\} \cup \{G, E\}$$
$$\{0, 6\} \cup -1 + \{0, 3\} \;\mathcal{Z}\; \{0, 6\} \cup -1 + CS_{\{3\}}(\{0, 3\})$$

or, as a transformation (using Z_{CS} here to indicate the z-transformation based on CS):

$$CZ = Z_{CS\{3\}}(CX) = Z_I(CX), \text{ or}$$
$$CX = Z_{CS\{3\}}(CZ) = Z_I(CX)$$

The cyclic-sub z-transformation between the sets CX and CZ is representative of one of two possible cyclic-sub z-transformations for the all-interval tetrachords, since the remainder set [03] can map onto another instance of [03] by transposing either {0} by T_6 (i.e. $CS_{\{0\}}$), or by transposing {3} by T_6 (i.e. $CS_{\{3\}}$). The two cyclic-sub z-transformations hold either [016] in common, or [026] in common. In the case of CX and CZ, a [016] subset is held in common.

4.5 Z-transformations and twelve-tone music

Since the z-transformations provide information concerning the relationship between z-related hexachords, it naturally follows that the z-transformations can be used to explain musical processes in twelve-tone music based on rows with two z-related hexachords. In contrast to the large body of the music-theoretical literature concerning twelve-tone music, which has focused heavily on hexachordal combinatoriality—a tradition already begun with Babbitt—the present theory offers some insights into works based on rows that specifically have z-related hexachords, which for the most part are not combinatorial.[2] Though they are not necessarily the majority (at least for Schoenberg and Webern, as well as many of the second generation of serial composers such as Pierre Boulez and Karlheinz Stockhausen), there are nevertheless a significant number of twelve-tone works that use rows with z-related hexachords.

In one particular twelve-tone work that uses a row with z-related hexachords, Schoenberg's 3rd String Quartet, op. 30, the properties of the z-relation are utilized for interrelating various row forms, by way of partial combinatoriality, row-segment invariance and row overlap. The quartet has a row that is ordered in a way that aligns with subset-union properties of the z-related hexachords.[3] The alignment between the

---

[2] See, for example, the table in Straus 2000, p. 224, which shows which hexachords are P, R, I or RI combinatorial.

[3] Martha Hyde (1980) showed using sketches that there are in fact three distinct rows in Schoenberg's 3rd String Quartet. In Schoenberg's sketches, the three rows are displayed as a chart (where each row is divided into a 5+2+5 partition) that indicates that the three rows share the first five pitch classes, but differ in the latter seven. For

ordering of the row and the structure of the z-relation between the hexachords allows for particular invariance relationships between different row forms, some of which Schoenberg clearly exploits throughout the quartet. With this particular row, invariances manifest via the dyadic segmentation of the row (order positions 0-1, 2-3, 4-5, etc.), which Schoenberg presents as a fundamental motivic idea at the beginning of the second movement (see Example 4.15), as a series of vertical dyads.

Example 4.15: Opening measures of Schoenberg's 3rd String Quartet, op. 30, II. Numbers above the pitches indicate order positions.

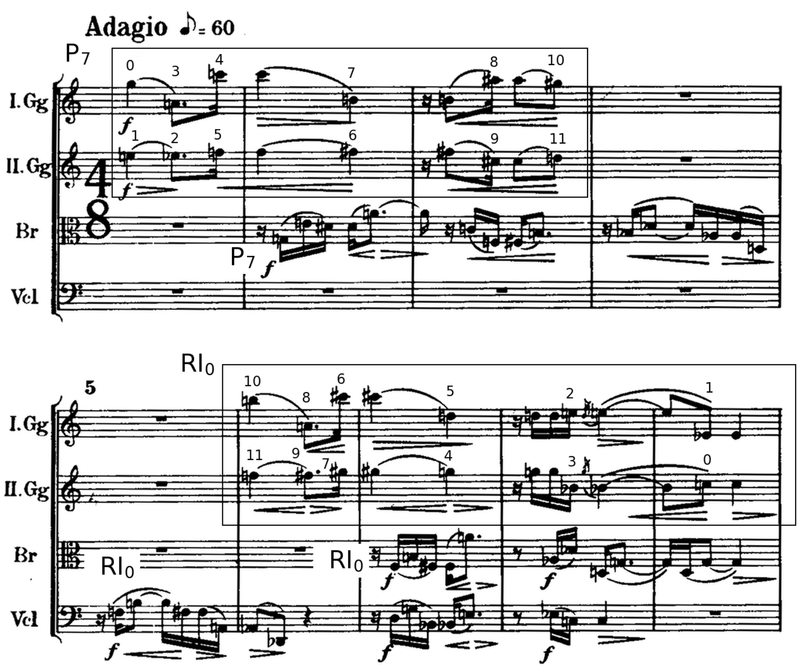


Table 4.7 shows the row from the quartet (in the middle the table), along with the z-related hexachord set classes (top row), and the dyad setclasses (bottom row).

---

the present purposes, I will discuss only the properties of the first of Hyde's three rows, which is more prominently featured in the quartet than are the other two rows.

As we can see, the ordering of the row parses both of the two z-related hexachords into the set classes [03], [05] and [06].

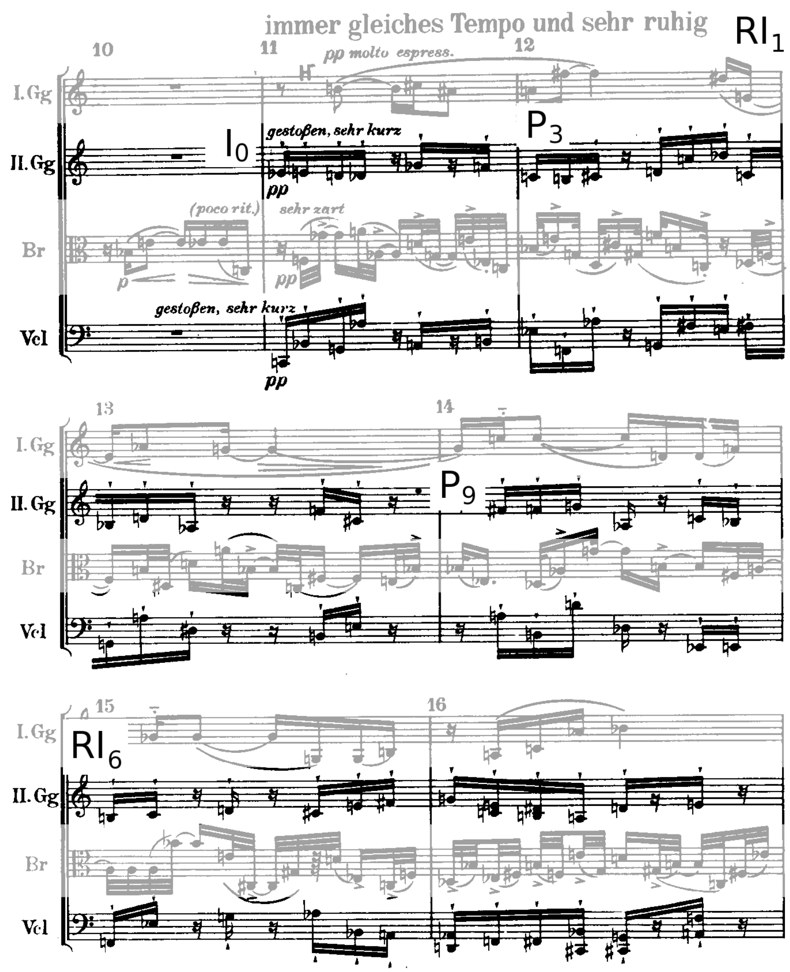
Table 4.7: The row of Schoenberg's 3rd String Quartet, and its hexachordal and dyadic partitions.

hexachords	[012469]						[013468]					
row	7	4	3	9	0	5	6	11	10	1	8	2
dyads	[03]		[06]		[05]		[05]		[03]		[06]	

The dyadic segmentation of the row is not only one of the core thematic features of the quartet, but it is also the conduit through which invariances between row forms are created. These invariance relationships are based on the cyclic-sub z-transformation and on pumping, both of which rely on the dyadic partition of the row (in that they only alter one or two of the dyads within each hexachord, leaving the others unchanged). Since the ordering of the row aligns with the dyadic structure of the z-related hexachords, the z-transformations parse the 48 row forms into a series of 6 row families with row forms that share 6 dyads.

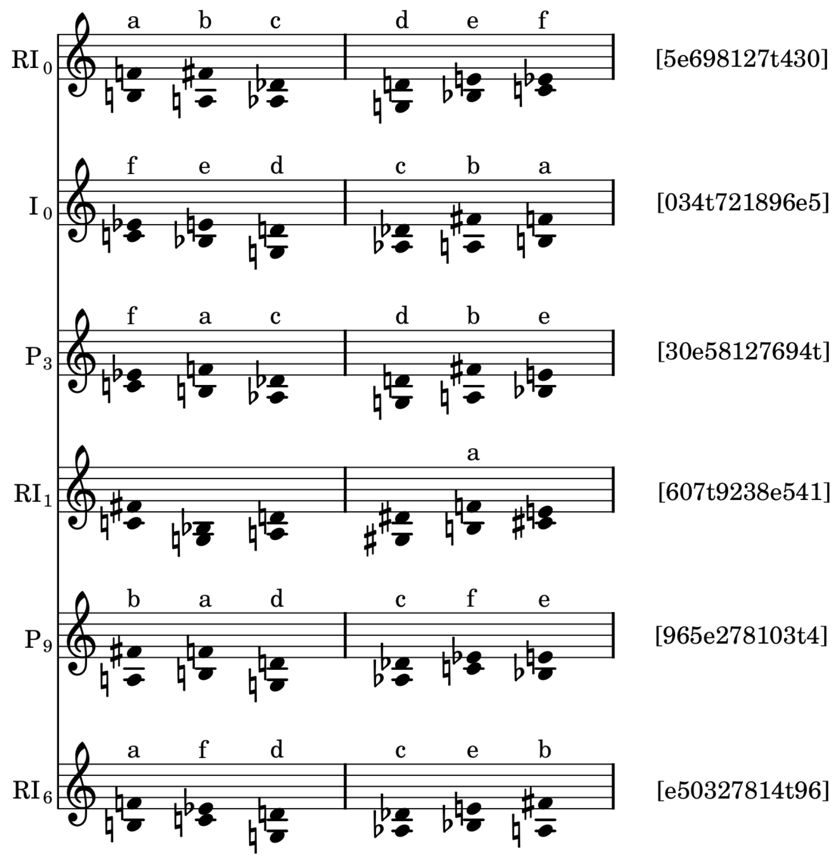
Before we look at the row-family structure based on z-transformations, let us consider the passage that follows the opening of the second movement (see Example 4.16), focusing particularly on what is happening with the 2nd violin and cello. Continuing the idea from the opening passage, this passage has two instruments (Vln II and Vcl) that are rhythmically synchronized. The two parts create a chain of vertical dyads that unfold 5 row forms (labeled in the example). For all but one row form, the various row forms in the passage have the same 6 dyads, though ordered

Example 4.16: Schoenberg, 3rd String Quartet, op. 30, II, mm. 10-16.



differently. The registers in which the pitch classes appear vary only somewhat, as a large number of the dyads are in fact repeated in the same register. Regardless, the repetition of the dyadic pitch-class sets is strictly maintained in this passage by the selection of row forms that share the same dyadic partition. To show this, Example 4.17 lists these five row forms and their dyadic partitions, beginning with the last row form from the opening passage (RI_0). As the example shows, all but one of the rows (RI_1) shares the same six dyadic pairs.

Example 4.17: Row forms presented as vertical dyads in Schoenberg's 3rd String Quartet, op. 30, II, mm. 5-15.



The one row that does not share the same dyadic partition with the other rows (RI_1) appears to fit into the chain differently. Rather than sharing dyads, the row form RI_1 begins with a hexachord that shares 5 pitch classes of the 2nd hexachord of the preceding row (P_3), sharing pitch classes F♯, G, A, B♭ and D. However, since the 2nd hexachord of RI_1 does not relate in the same manner to the following row (P_9), this explanation is perfunctory at best. Nevertheless, because of the consistent appearance of dyadic relationship between row forms, in this passage as well as others, it is

useful to consider the row in terms of its dyadic partitions and the resulting row family structure.

The various row forms that share the same dyads group into row families according to the z-transformations applied to the two hexachords (a and b). Each of the rows in such a family is related to the others by the z-transformations $Z_{CS[03]}$ (the z-transformation that transposes the [03] subset by 6), $Z_{CS[05]}$ (transposes the [05] by 6), or Z_T* (transposes the [06] by 6), by T_6, or by some combination of those. In total, the row forms in the family are related by eight transformations (including the identity X), which together form a transformation group (with closure, associativity, identity and inversibility). Table 4.8 is the combination table for the group of transformations.

Table 4.8: Combination table for the transformations among the row forms in the family.

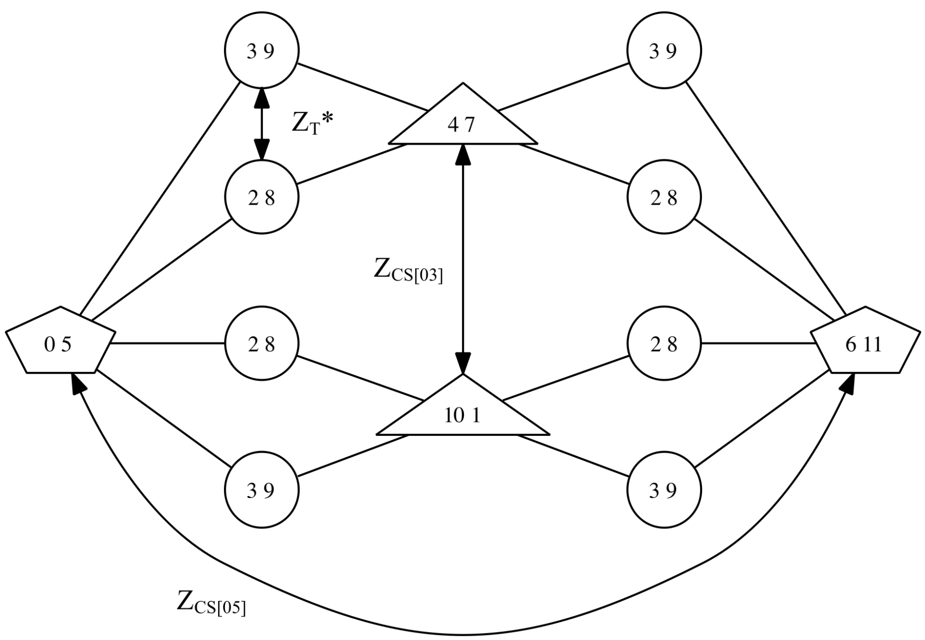
	X	T_6	Z_T*	T_6 Z_T*	$Z_{CS[03]}$	$Z_{CS[05]}$	$Z_{CS[03]}Z_T$*	$Z_{CS[05]}Z_T$*
X	X	T_6	Z_T*	T_6Z_T*	$Z_{CS[03]}$	$Z_{CS[05]}$	$Z_{CS[03]}Z_T$*	$Z_{CS[05]}Z_T$*
T_6	T_6	X	T_6Z_T*	Z_T*	$Z_{CS[05]}$	$Z_{CS[03]}$	$Z_{CS[05]}Z_T$*	$Z_{CS[03]}Z_T$*
Z_T*	Z_T*	T_6Z_T*	X	T_6	$Z_{CS[03]}Z_T$*	$Z_{CS[05]}Z_T$*	$Z_{CS[03]}$	$Z_{CS[05]}$
T_6Z_T*	T_6Z_T*	Z_T*	T_6	X	$Z_{CS[05]}Z_T$*	$Z_{CS[03]}Z_T$*	$Z_{CS[05]}$	$Z_{CS[03]}$
$Z_{CS[03]}$	$Z_{CS[03]}$	$Z_{CS[05]}$	$Z_{CS[03]}Z_T$*	$Z_{CS[05]}Z_T$*	X	T_6	Z_T*	T_6Z_T*
$Z_{CS[05]}$	$Z_{CS[05]}$	$Z_{CS[03]}$	$Z_{CS[05]}Z_T$*	$Z_{CS[03]}Z_T$*	T_6	X	T_6Z_T*	Z_T*
$Z_{CS[03]}Z_T$*	$Z_{CS[03]}Z_T$*	$Z_{CS[05]}Z_T$*	$Z_{CS[03]}$	$Z_{CS[05]}$	Z_T*	T_6Z_T*	X	T_6
$Z_{CS[05]}Z_T$*	$Z_{CS[05]}Z_T$*	$Z_{CS[03]}Z_T$*	$Z_{CS[05]}$	$Z_{CS[03]}$	T_6Z_T*	Z_T*	T_6	X

For an example of a row family, let us look at the row family that includes P_7, the initial row of the second movement (see Table 4.9).

Table 4.9: A row family based on the z-transformations, with transformations in relation to P_7.

P_7	7	4	3	9	0	5	6	11	10	1	8	2	-
R_7	2	8	1	10	11	6	5	0	9	3	4	7	$T_6 Z_T{}^*$
RI_4	9	3	10	1	0	5	6	11	2	8	7	4	$Z_{CS[03]}$
I_4	4	7	2	8	11	6	5	0	1	10	3	9	$Z_{CS[05]} Z_T{}^*$
RI_{10}	3	9	4	7	6	11	0	5	8	2	1	10	$Z_{CS[05]}$
I_{10}	10	1	2	8	5	0	11	6	7	4	9	3	$Z_{CS[03]} Z_T{}^*$
P_1	1	10	9	3	6	11	0	5	4	7	2	8	T_6
R_1	8	2	7	4	5	0	11	6	3	9	10	1	$Z_T{}^*$

Example 4.18: Z-transformation network for the z-related hexachordal pair of the row from Schoenberg, 3rd String Quartet (modeled after the CUP wreaths in Morris 1990).



For each of the row forms in the table, the right-most column indicates the relationship of the row form to P_7: that is, applying the indicated transformation to the two hexachords of P_7 yields the two hexachords of the row form. Overall, all eight

row forms in the family have the same discrete dyadic partition, though the dyads are in different locations and sometimes have switched pitch-class orderings. Despite the changes in ordering, the common dyadic partitions allow us to represent the eight row forms within the dyad-union network, shown in Example 4.18

In the network, any combination of adjacent pentagons, triangles and circles with exactly one of each shape forms a hexachord of two of the eight row forms in the family (the two row forms being retrogrades of each other). The remaining pentagon and triangle, and the other circle that lies in between (either {2, 8} or {3, 9}), form the complementary hexachord. The lines with arrows show how the nodes of the graph are affected by the z-transformations: $Z_{CS[03]}$ changes the triangles, $Z_{CS[05]}$ the pentagons, and Z_T* the circles.

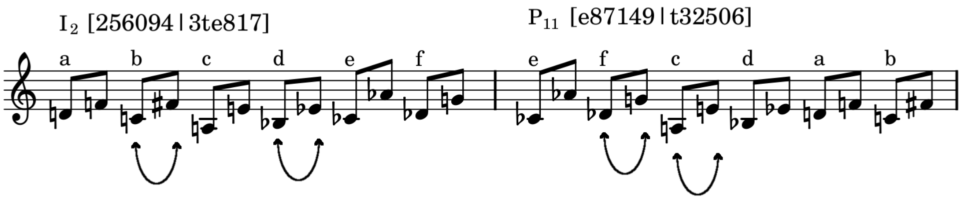
While the network maintains the dyadic partition constant among the row forms in the family, the ordering of the individual dyads is left undetermined. Likewise, in the 3rd String Quartet, there are multiple cases where the orderings of the individual dyads are masked by being presented as vertical dyads, as we saw in Examples 4.15 and 4.16. There is even one passage in the quartet (see Example 4.19) where the orderings of the pitches within the dyads are switched (even if only visually), creating a larger invariance among the discrete tetrachords of two row forms.

Example 4.19: Schoenberg, 3rd String Quartet, II, mm. 41-43.

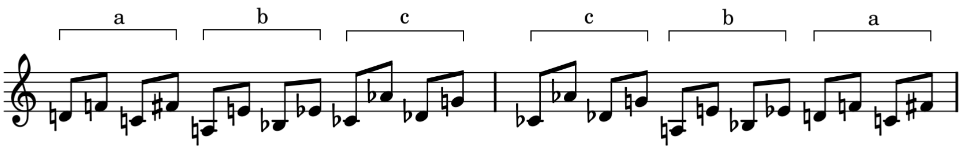
In this passage, the second violin has two successive rows: I_2 in mm. 41–42$^{1/2}$ and P_{11} in mm. 42$^{1/2}$–43. The two rows are each presented as a series of six tremolo dyads. If we are to read the tremolo dyads as effectively standing in as vertical dyads, then there is no problem with the ordering of the pitches within each dyad, since it is undetermined. Otherwise, if we read the ordering as it is on the page (with the tremolo starting on the lowest note), we find that for some of the dyads the ordering of the two pitch-classes are switched (see Example 4.20a). Perhaps this reordering is merely a byproduct of the standard tremolo notation; it is also arguable, however, that the violinist would, in fact, physically perform the tremolo starting on the lower note. Nevertheless, as a consequence of the pitch reordering, the dyads of the two row forms now pair up, so that the two row forms share identical discrete tetrachords with the same pitch orderings (see Example 4.20b). Between the two rows, the tetrachordal partitions are the same, except that the tetrachords are ordered in retrograde in respect to one another (abc vs. cba).

Example 4.20: Reordered dyads in two successive row forms in Schoenberg, op. 30, II, mm. 41-43.

a)



b)



Even if we consider the tremoli to represent vertical dyads, and not reordered dyads, the registral placement of the pitches strongly emphasizes the shared dyads between the rows, revealing a correspondence between both the dyadic and tetrachordal partitions of the two row forms.

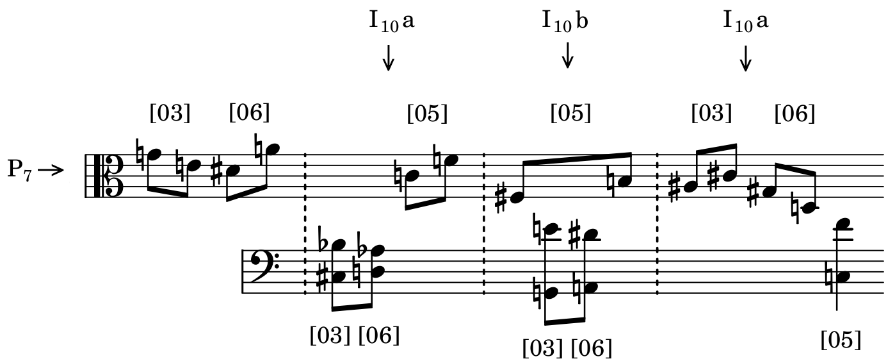
So far we have looked at row families based on z-transformations. We can also consider row families that are based on pumping. The beginning of the third movement of the string quartet, for instance, is a case where it is perhaps better to understand the invariances between row forms in terms of pumping instead of the z-transformations discussed above. In this passage (see Example 4.21) row forms overlap, either by sharing a [0146] all-interval tetrachord or a [05] dyad.

Example 4.21: Schoenberg, 3rd String Quartet, III, mm. 1-5.

In these first six measure of the third movement, there are two layers of row forms: 1) the viola has complete rows (P_7 in mm. $0–3^{1/2}$ and I_0 in mm. $3^{1/2}–6$), while 2) the violin II and cello interject once per measure, making hexachords with some of the notes in the viola. In mm. 1, 2, 4 and 5, the violin II and cello interject with all-interval tetrachords, which form hexachords along with the [05] dyads in the viola line. In m. 3, they interject with [05] dyads, forming hexachords with the [0146] all-interval tetrachords in the viola line. Example 4.22 highlights these relationships in mm. 0–3.

Example 4.22: Overlapping row form based on the partition with the [0146] all-interval tetrachord; Schoenberg, 3rd String Quartet, III, mm. 0-3.

P_7	7	4	3	9	0	5	6	e	t	1	8	2
I_{10}	t	1	2	8	5	0	t	6	7	4	9	3
subsets	[0146]				[05]		[05]		[0146]			



As was shown above in Table 4.9, the row forms P_7 and I_{10} are related by $Z_{CS[03]}Z_T$*, meaning that between the two row forms the [03] subsets in each of the hexachords is replaced by T_6 of the [03] subset, and the [06] subset is replaced by $T_{\pm 1}$ of the [06] subset. However, the two row forms are also related by pumping: the pair is generated by pumping the [0146] all-interval tetrachord with the pumping algorithm *p*5a—the algorithm that generates z-related sets starting with only one of the all-interval tetrachords. For this particular hexachordal pair, pumping two [0146]s with *p*5 is the only way to generate the pair using pumping algorithms.

To see how this pumping-based relationship among the hexachords of the various row forms would be expressed, let us briefly review pumping algorithm *p*5a. In Chapter 3, algorithm *p*5a was defined as: Let $\Phi = \{0, 6\}$ and $\Psi = \{0, 3\}$, and use a formula where the two sides are equal.

$$\Phi \uplus x + \Psi \qquad = \qquad \Phi \uplus x + \Psi$$

Let $i = 2x - 3$, and S be a set that is symmetrical at index i. Add S to one side of the formula, and $i + 6 - S$ to the other.

$$\Phi \uplus x + \Psi \uplus S \qquad \mathcal{Z} \qquad \Phi \uplus x + \Psi \uplus i + 6 - S$$

With a little reverse engineering, we can find the formula that generates the two z-related hexachordal set classes of the row. We find that instances of the z-related set classes are formed by setting the value $x = 1$ and the set $S = \{3, 8\}$ (which is symmetrical at index $i = -1$):

$$\{0, 6\} \cup 1 + \{0, 3\} \cup \{3, 8\} \quad \mathcal{Z} \quad \{0, 6\} \cup 1 + \{0, 3\} \cup 5 - \{3, 8\}$$
$$\{0, 1, 3, 4, 6, 8\} \quad \mathcal{Z} \quad \{0, 1, 2, 4, 6, 9\}$$

This pumped formula shows that there is relationship between z-related hexachords based on the union of an [0146] all-interval tetrachord and a [05] dyad, which is precisely the relationship that manifests in mm. 2–3 of the example above: the second hexachord of P_7 overlaps with the first hexachord of I_{10}, sharing a common [0146] all-interval tetrachord. The two hexachords (P_7b and I_{10}a), then, are simply some transposition and/or inversion of the two sets produced by the algorithm. Specifically, the two hexachords (P_7b and I_{10}a) are related to the two hexachords produced by the algorithm by the inversion I_2.

P_7b: 6 e t 1 8 2
I_{10}a: t 1 2 8 5 0

$$P_7\text{b} \quad \mathcal{Z} \quad I_{10}\text{a}$$
$$2 - \{0, 1, 3, 4, 6, 8\} \quad \mathcal{Z} \quad 2 - \{0, 1, 2, 4, 6, 9\}$$

Thus, while the two hexachords (P_7b and $I_{10}a$) both share the all-interval tetrachord {t, 1, 2, 8}, they have [05] subsets ({6, e} and {5, 0}) that are related by inversion at index 2–5+2 = –1.[4]

As has been shown using example from Schoenberg's 3rd String Quartet, the properties of z-related hexachords can be exploited to create invariance relationships between different row forms. Such invariances arise when the row is ordered so that it is in alignment with the subset unions that generate the z-related hexachords. Depending on how the relationships manifest in the music, we may want to describe the relationships in terms of the z-transformations (Z_T, Z_I and Z_{CS}), or in terms of pumping, both of which describe the z-related hexachords as being formed by common or z-related subsets. Schoenberg's 3rd String Quartet provides an instance of where both the z-transformations and pumping could be use to describe the underlying invariance relationships that appear in the music. Ultimately, the examples from the quartet show that the relationships described here are relevant to the analysis of twelve-tone music, and that there are potentially other undiscovered twelve-tone compositions that similarly exploit the properties of z-related sets.

4.6 Using the Fourier partition to identify z-related sets

So far we have seen some ways in which z-transformation can be used to investigate z-related pitch structure. Now I want to change directions, and discuss for a moment the Fourier partition as a means for identifying z-related sets.

---

[4] This follows from the fact that if B = I_xA, then the sets I_yA and I_yB will be related by I_{y-x+y}.

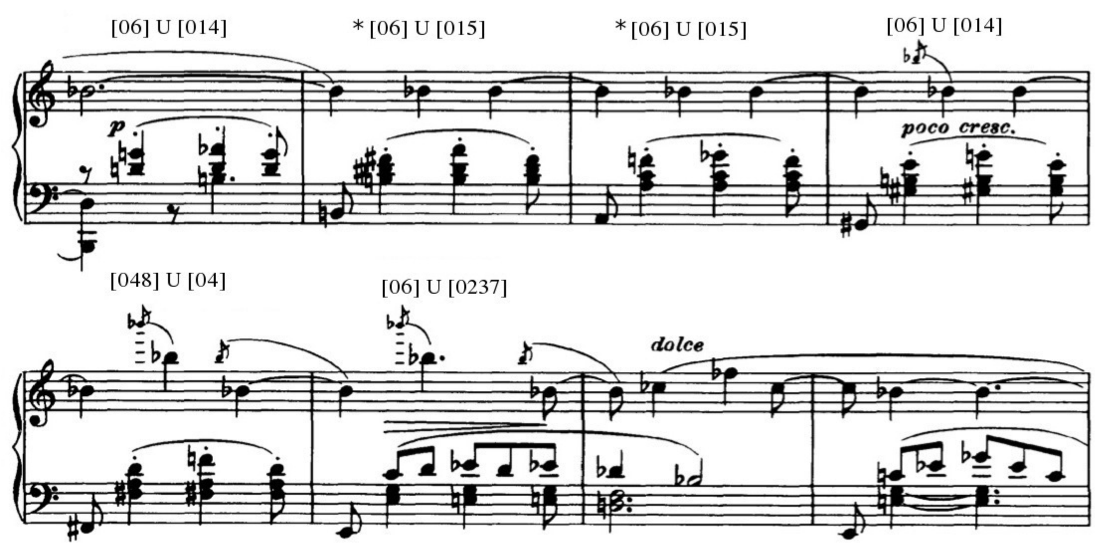
A common style of pitch-class-set analysis begins with identifying certain set classes that are potentially relevant to the given piece (either as motives or harmonies), and to work through the piece looking for instances of those set classes. This kind of analysis requires a fair amount of drudgework by the analyst. In practice, the cardinality of the set class is a big factor as the to amount of work needed to scan the piece for set classes. Looking for particular intervals (dyads), for example, is much easier than looking for hexachords. For larger sets, however, this process can be speed up by using particular identifying features of the set classes (like a [012] cluster, or a [0134] octatonic fragment).

The present theory demonstrates that all of the z-related sets in mod12 share such a defining feature: being the union of cyclic collections plus a remainder set, the sets of a z-related pair always share the discrete partition (the Fourier partition) that extracts all the cyclic collections. This feature, I argue, facilitates the identification of z-related pairs by providing a heuristic that easily determines whether or not a given pitch-class set is z-related. Instead of considering sets as single objects with many pitch classes, one only needs to consider z-related sets as unions of cyclic collections. Not only is it an aid for the identification of z-related sets, but the Fourier partition also provides important information about set-class identity and complementation. Overall, the Fourier partition gives a strikingly accurate set-class profile of sets, all while requiring minimal effort.

Let us consider a couple examples to see how the Fourier transform might be used for set-class/z-related-set identification. Example 4.23 shows mm. 10-17 of the

final movement of Bartòk's Suite for Piano, Op. 14, which is a passage that Allen Forte cited as having an instance of the z-relation.[5] The passage consists of series of chords (each lasting one measure) that move over a pedal B♭ in the upper-most voice. Each of the chords in mm. 10-14 (including the B♭) is a five-note chord, and each contains a triad. Overall, the underlying pattern in these measures involves a descent of the triad by step, where in each measure the upper voice of each triad moves to a neighbor tone and back.

Example 4.23: Bartòk, Suite for Piano, Op. 14, IV (Sostenuto), mm. 10-17. Z-relation and set-class identification using the Fourier partition.



m. 10:	{D, A♭} ∪ {G, B♭, B}	=> [06]+[014]
m. 11:	{D♯, A} ∪ {F♯, B♭, B}	=> [06]+[015]
m. 12:	{C, G♭} ∪ {F, A, B♭}	=> [06]+[015]
m. 13:	{E, B♭} ∪ {G, G♯, B}	=> [06]+[014]
m. 14:	{D, F♯, B♭} ∪ {F, A}	=> [048]+[04]
m. 15:	{E, B♭} ∪ {C, D, E♭, G}	=> [06]+[0237]

[5] Forte 1973, p. 24.

The bottom of Example 4.23 provides the Fourier partitions for each of the chords. As shown, the first four chords partition into a [06] plus a set, which is either [014] or [015]. The chords in mm. 14-15 are different from the previous four measures: the chord in m. 14 has a Fourier partition with a [048] subset; the chord in m. 15 is a 6-note chord with a 4-note remainder set.

Let us consider briefly the Fourier partitions of the various chords in mm. 10-14 to determine if any of them are of the same set class or z-related. From our investigations on the z-relation, we know that the chords in mm. 10 and 13 cannot form a z-related pair even though they share the same remainder set, since [014] does not have cyclic invariance in respect to the 6-cycle, and thus is not a valid remainder set. Upon further inspection, however, we find that they are of the same set class, since in both cases the [014] is situated similarly in respect to the [06] cycle, with the two chords being related by I_6. The two chords in mm. 11-12 also have identical remainder sets ([015]), which is indeed a valid remainder set in respect to the 6-cyclic (that is, it possesses cyclic invariance). Since all possible non-intersecting unions of [06] and [015] form only the two sets of the pentachordal z-related pair, these two chords in mm. 11-12 must either be z-related or of the same set class. With further inspection, we find that they are in fact z-related, since the chord in m. 11 has a [012] cluster—A, B♭, B—while the chord in m. 12 does not.

The chord in m. 14 is unrelated to the chords in mm. 10-13 and m. 15, being instead based on the [048] cyclic collection. Furthermore, its remainder set is not a valid remainder set in respect to the [048] cycle, which indicates that this chord has

no potential z-related partners. The chord in m. 15 is a 6-note chord, and therefore cannot be z-related or of the same set class as the other five chords; however, the remainder set [0237] contains one [015] subset, and thus the chord is a superset of one of the two z-related pentachords in mm. 11-12. Upon further inspection, we find that the chord in m. 15 is in fact an abstract superset of the chord in m. 11 ([01258]), the set of the z-related pair that has a [012] subset. In this particular passage, however, the superset relationship between m. 15 and m. 11 does not align with the underlying process of the passage, the process in which triads move against the pedal B♭—in m. 15 the [01258] subset is everything but the pitch C♮, but the pitch D would be the "extra" pitch, being a passing tone between the highest note of the triad (C) and the upper neighbor (E♭). In other words, though the chord in m. 15 abstractly includes the chord in m. 11, the segmentation that parses out the [01258] subset in m. 15 does not align with the surface features of the chord (emphasizing D instead of C); consequently, the particular subset/superset relationship between mm. 11 and 15 would not be considered as particularly significant.

Looking first at the Fourier partitions gives a quick picture of how the sets are related. In Example 4.23, the Fourier partition captures set-class identity as well as the z-relation. However, it is also possible to use the Fourier partition to identify complementary sets. As shown before in Chapters 2 and 3, complements have the same Fourier partition (with the same remainder set) as do their partners, except that the complement has addition cyclic collections in the partition—a fact that holds true for all complementary pairs in mod12. Thus, by examining the Fourier partition, we

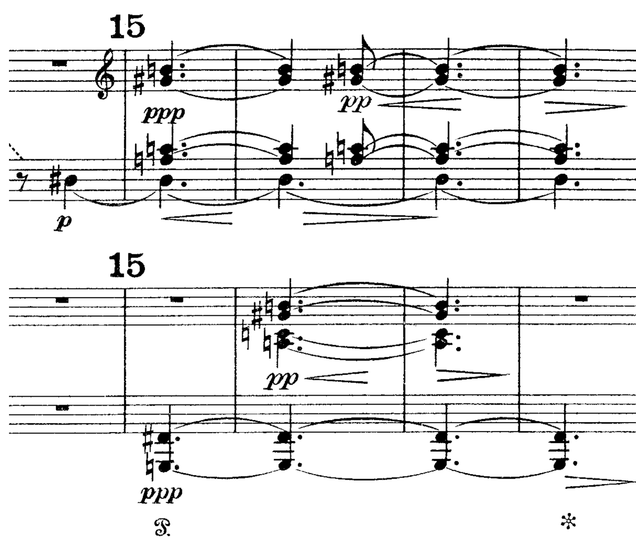
can quickly narrow down whether or not a pair of given sets are potentially complementary.

Take, for example, the final chord of the ‘Das obligate Rezitativ’ from the final movement of Schoenberg’s Fünf Orchesterstücke, Op. 16 (shown as Example 4.24). The upper-part of the chord (played by piano I at rehearsal 15) is a pentachord that has a Fourier partition with one instance of [06] and one instance of [014]:

{D♯, A} ∪ {G♯, B, C} => [06]+[014]

The partition is emphasized by the particular voicing, placing the tritone below the [014] trichord.

Example 4.24: The final chord of Schoenberg’s Fünf Orchesterstücke, Op. 16, v (arranged for 2 pianos by Anton Webern).



The complete chord including the two bass notes in piano II (the G and F♯) is a heptachord that also has a Fourier partition with the remainder [014], but that has as well an additional instance of [06]:

{D♯, A} ∪ {F♯, C} ∪ {G, G♯, B} => [06]+[06]+[014]

Adding the G and F♯ (an instance of [01]) to the chord maintains the same remainder set, since though the C becomes involved in the second [06] subset, it is replaced by the G to form an [014] along with G♯ and B. Though in this case having identical remainder sets in the Fourier partition does not guarantee that these two sets are abstract complements—since there are three unique set classes formed by a non-overlapping union of [06] and [014], to any of which the 7-note set could be the complement—the Fourier partition is at least a partial indicator of whether or not these two sets are complementary.

The Fourier partition is an aid for identifying set classes, z-related sets and complements. For the identification of z-related sets, the Fourier partition is the first of two steps: first, a given pair of sets must have the same Fourier partition, with the same cyclic collections and remainder set; and second, the remainder set must have cyclic invariance in respect to the cyclic collection. Using the Fourier partition to gather preliminary information on sets greatly reduces the time needed to determine whether or not a pair of set classes is z-related (or of the same set class), while at the same time staying close to the generative principles that produce the z-relation in the first place.

Conclusion

Using methods inspired by research in crystallography, I have developed a number of formulas for showing subset relations shared by z-related sets. The formulas express the z-related sets using only simple algebra, showing that between two z-related sets there is a positive/negative relation among the subset positions within the two sets—both for z-pairs with one cyclic-collection subset and those with multiple cyclic-collection subsets. This aspect of the z-relation allows us to conceptualize z-transformations simply as a shift from one side of the formula to the other, where one subset is held in common and the others are transposed or inverted.

The formulas not only show the relationships between z-related sets, but they are also generative. Using the criteria that I provided for remainder sets, it is possible to create (in theory, at least) an infinite number (though not all) of the z-related sets in the various moduli, including all of the 23 z-related pairs in mod12. However, since my formulas do not cover all z-related sets in all moduli, they cannot enumerate the z-related sets—a task that, if even possible, would instead require formulas that involve polynomials, as demonstrated by Rosenblatt (1984).

A major advantage in defining z-transformations in terms of positive/negative relationships between the positions of the subsets is that the transformations participate in a group structure, which links together all the instances of the two z-related set classes that share the cyclic collection. Even though the z-transformations are contextual transformations, they enjoy the same group properties that we find with some of the more common transformations, such as commutativity, associativity

and the inverse. The z-transformations thus combine well with the usual set operators, and do not disrupt the overall group structure of the operations.

In addition to the formulas, and their consequent z-transformations, I have shown that the Fourier transform provides specific information about the subset content of z-related sets. The Fourier transform reveals that the cyclic-collection content of the sets of z-related pair is always the same, whether or not the remainder sets are of the same set class. Thus, except for the z-related sets that have no cyclic-collection subset, such as the z-related tetrachord in mod13, we see that the cyclic-collection plays a vital role in how the z-relation comes about.

Overall, my work subsumes the work of both Morris and Soderberg, as well as offshoots (such as Capuzzo), who had developed the most thorough explanations of the z-relation before music theorists became aware of the corresponding work in the field of crystallography on homometric sets. Not only does my work reveal the underlying principles on which the z-relation is based, but it simplifies the algebra of Morris and Soderberg, allowing the theory to be much more practical for music composition and/or analysis.

Though while my vision has been to use this information to analyze pitch structures in 20th-century music, the theory could also be used to analyze other features, such as rhythm. Furthermore, the theory extends to other modular spaces besides mod12, and thus could be used to analyze set structures in, for example, quarter-tone space. In short, the breadth of the theory allows it to be applied to much

more than just the all-interval tetrachords or z-related hexachords in 12-tone chromatic space.

The formulas also welcome a change in perspective on the nature of z-related sets. While traditionally the z-relation has simply indicated that two sets share the same interval vectors, the present theory shows that z-related sets also share particular pitch-class structures. The benefit here is that now instead of having to refer only to some abstract property of the set (the interval-class content), we can now consider concrete aspects of the z-relation. This shift is crucial for the usefulness of the z-relation as a musical concept, since it reorients our attention from an aspect that is mostly intangible to other aspects that are concretely manifested in the pitches themselves. With a focus on the concrete aspects of the z-relation, certain tasks that would otherwise be overwhelming become quite manageable, such as, for example, improvisation with z-related hexachords.

Some of the theory presented here also has applications beyond the topic of the z-relation. My work on the Fourier transform, for instance, could be extended to pitch-class sets. The notion that sets with equal-magnitude and opposite-pointing $\mathbb{F}_x$ coefficients form supersets that have FOURPROP(x) remains true whether we are considering z-related sets or other non-z-related sets. Furthermore, I showed that the Fourier partition is useful for the identification of sets that are z-related, as well as sets that are of the same set class or complementary. Though I presented these ideas mainly in reference to the z-relation, these aspects of my theory indeed reflect on

features of the pitch-class universe on the whole, and thus could be extended to other yet considered areas of pitch-class set theory.

Overall, the theory presented here provides a comprehensive framework for the analysis of z-related pitch-class sets. Though it is impossible to foresee all of its future applications, I am confident that the theory provides a useful starting point for a variety of types of analysis.

Bibliography

Amiot, Emmanuel. 2007. David Lewin and Maximally Even Sets. *Journal of Mathematics and Music* 1/3: 157-172.

Babbitt, Milton. 1955. Some Aspects of Twelve-Tone Composition. *The Score and IMA Magazine* 12: 53-61.

__________. 1992. *The Function of Set Structure in Twelve-Tone Music* [1946]. PhD Dissertation, Princeton University.

Bagchi, S. N. and R. Hosemann. 1954. On Homometric Structures. *Acta Crystallographica* 7: 237-241.

Ballinger, et al. 2009. The Continuous Hexachordal Theorem. In Mathematics and Computation in Music, Volume 38 of *Communications in Computer and Information Science*.

Bernard, Jonathan. 1997. Chord, Collection, and Set in Twentieth-Century Theory. In J. Baker, D. Beach, and J. Bernard (Eds.), *Music Theory in Concept and Practice*. Rochester, NY: University of Rochester Press.

Brinkman, Alexander R. 1986. A Binomial Representation of Pitch for Computer Processing of Musical Data. *Music Theory Spectrum* 8: 44-57.

__________. 1990. *Pascal Programming for Music Research.* Chicago: University of Chicago Press.

Browne, Richmond. 1974. Review of Allen Forte's Theory of Chords. *Journal of Music Theory* 18/2: 390-415.

Buerger, M. J. 1950. Vector sets. *Acta Crystallographica* 3: 87-97.

__________. 1959. *Vector Space*. New York: John Wiley & Sons.

__________. 1962. The Algebra and Geometry of Convolutions. *Atti. Acad. Noz. Jincei. Mem. Cl. Sci. Math. Nat. Sez*. 11/8: 683-695.

__________. 1975. Proofs and Generalizations of Patterson's Theorems on Homometric Complementary Sets. *Z. Kristallogr*. 143: 79-98.

__________. Interpoint Distances in Cyclotomic Sets. *Can. Mineral*. 16: 301-314.

Bullough, R. K. 1961. On Homometric Sets. I. Some General Theorems. *Acta Crystallographica* 14: 257-268.

__________. 1964. On Homometric Sets. II. Sets Obtained by Singular Transformations. *Acta Crystallographica* 17: 295-308.

Callender, Clifton. 2007. Continuous Harmonic Spaces. *Journal of Music Theory* 51/2: 277-332.

Callender, Clifton and Rachel Hall. 2008. Crystallography and the structure of z-related sets. Paper given at the annual meeting of the Society for Music Theory in Nashville, TN. Handout accessed from myweb.fsu.edu/ccallender/z-relationhandout.pdf on Jan. 3, 2012.

Capuzzo, Guy. 2000. *Variety within unity: Expressive means and their technical ends in the music of Elliott Carter*, 1983-1994. PhD Dissertation, The University of Rochester, Eastman School of Music.

__________. 2004. The Complement Union Property in the Music of Elliott Carter. *Journal of Music Theory* 48/1: 1-24.

Carter, Elliott. 2002. *Harmony Book*. Ed. by Nicholas Hopkins and John F. Link. New York: Carl Fischer.

Childs, Adrian. 2006. Structural and Transformational Properties of All-Interval Tetrachords. *Music Theory Online* 12/4.

Clough, John. 1965. Pitch-Set Equivalence and Inclusion (A comment on Forte's Theory of Set-Complexes). *Journal of Music Theory* 9/1: 163-180.

Cohen, Allen Laurence. 2004. *Howard Hanson in Theory and Practice.* Westport, CT: Greenwood Press.

Forte, Allen. 1964. A Theory of Set-Complexes for Music. *Journal of Music Theory* 8/2: 136-183.

__________. 1965. The Domain and Relations of Set-Complex Theory. *Journal of Music Theory* 9: 173-80.

__________. 1973. *The Structure of Atonal Music*. New Haven: Yale University Press.

Hanson, Howard. 1960. *The Harmonic Materials of Twentieth-Century Music*. New York: Appleton-Century-Crofts.

Hasty, Chirstopher F. 1990. An Intervallic Definition of Set Class. *Journal of Music Theory* 31: 183-204.

Headlam, David. 1987. Review of 'The Operas of Alban Berg'. *In Theory Only* 9/5-6: 57-76.

__________. 1992. Fritz Heinrich Klein's "Die Grenze der Halbtonwelt" and "Die Maschine." *Theoria* 6: 55-96.

__________. 1996. *The Music of Alban Berg. Composers of the Twentieth Century.* New Haven: Yale University Press.

__________. 2006. Walter O'Connell's 'Tone Spaces': An Early View of the Order-Position / Pitch-Class Exchange. *University of Western Ontario Studies in Music* 19-20: 113-160.

Hook, Julian. 2007a. Review: David Lewin and the Complexity of the Beautiful. *Intégral* 21: 155-190.

__________. 2007b. Why are there Twenty-Nine Tetrachords? A Tutorial of Combinatorics and Enumeration in Music Theory. *Music Theory Online* 13/4.

Howe, Hubert S. 1965. Some Combinatorial Properties of Pitch Structures. *Perspectives of New Music* 4/1: 45-61.

Iglesias, Juan E. 1981. On Patterson's cyclotomic sets and how to count them. *Z. Krystallogr.* 156: 187-196.

Jenkins, J. Daniel. 2010. After the Harvest: Carter's Fifth String Quartet and the Late Late Style. *Music Theory Online* 16/3.

Kassler, Michael. 1961. *The Decision of Arnold Schoenberg's Twelve-Note-Class System and Related Systems*. Princeton, NJ: Princeton University Press.

Klein, Fritz Heinrich. 1925. Die Grenze der Halbtonwelt. *Die Musik* 17/4: 281-86.

Lewin, David. 1959. Intervallic Relations Between Two Collections of Notes. *Journal of Music Theory* 3/2: 298-301.

__________. 1960. The Intervallic Content of a Collection of Notes, Intervallic Relations between a Collection of Notes and its Complement: an Application to Schoenberg's Hexachordal Pieces. *Journal of Music Theory* 4/1: 98-101.

__________. 1976. On the Interval Content of Invertible Hexachords. *Journal of Music Theory* 20/2: 185-188.

__________. 1982. On Extended Z-Triples. *Theory and Practice* 7/1: 38-39.

__________. 1987. *Generalized Musical Intervals and Transformations*. New Haven: Yale University Press.

__________. 2001. Special Cases of the Interval Function between Pitch-Class Sets X and Y. *Journal of Music Theory* 45/1: 1-29.

Lewin, David, Forte, and Regener. 1977. Forte's "Interval Vector, My Interval Function", and Regener's "Common-Note Function". *Journal of Music Theory* 21/2: 194-237.

Mardix, S. 1990. Homometricism in close-packed structures. *Acta Crystallographica* A46: 133-138.

Martino, Donald. 1961. The Source Set and Its Aggregate Formations. *Journal of Music Theory* 5/2: 224-273.

Mead, Andrew. 1985. Large-Scale Strategy in Arnold Schoenberg's Twelve-Tone Music. *Perspectives of New Music* 24/1: 120-157.

Morris, Robert. 1979-80. A Similarity Index for Pitch-Class Sets. *Perspectives of New Music* 18/1-2: 445-60.

__________. 1982. Set Groups, Complementation, and Mappings among Pitch-Class Sets. *Journal of Music Theory* 26/1: 101-144.

__________. 1982-3. Combinatoriality without the Aggregate. *Perspectives of New Music* 21/1-2: 432-486.

__________. 1987. *Composition with Pitch-Classes: A Theory of Compositional Design*. New Haven: Yale University Press.

__________. 1990. Pitch-Class Complementation and Its Generalizations. *Journal of Music Theory* 34/2: 175-245.

__________. 2001. *Class Notes for Advanced Atonal Music Theory*. Frog Peak Music.

O'Connell, Walter. 1968. Tone Spaces. *Die Reihe* 8: 35-67.

O'Rourke, et al. 2008. A Pumping Lemma for Homometric Rhythms. Paper given at Montréal CCCG, August 11-13.

Patterson, A. L. 1939. Homometric Structures. *Nature* 143: 939-940.

__________. 1944. Ambiguities in the X-ray Analysis of Crystal Structure. *Phys. Rev.* 65: 195-201.

Pauling, Linus and M. D. Shappell. 1930. The Crystal Structure of Bixbyite and the C-modification of the Sesquioxides. *Z. Kristallogr.* A75: 128-142.

Perle, George. 1962. *Serial Composition and Atonality: An Introduction to the Music of Schoenberg, Berg, and Webern*. Berkeley: University of California Press.

__________. 1982. Letter from George Perle. *Journal of the American Musicological Society* 35/2: 373-377.

__________. 1990. Pitch-Class Set Analysis: An Evaluation. *The Journal of Musicology* 8/2: 151-172.

Quinn, Ian. 2001. Listening to Similarity Relations. *Perspectives of New Music* 39/2: 108-158.

__________. 2004. *A Unifed Theory of Chord Quality in Equal Temperaments*. PhD Dissertation, University of Rochester: Eastman School of Music.

__________. 2007. General Equal-Tempered Harmony (Parts II and III). *Perspectives of New Music* 45/1: 4-63.

Rahn, John. 1980. *Basic Atonal Theory*. New York: Schirmer Books.

Regener, Eric. 1974. On Allen Forte's Theory of Chords. *Perspectives of New Music* 13: 191-212.

Riordan, John. 1958. *An Introduction to Combinatorial Analysis.* New York: John Wiley & Sons.

Rosenblatt, Joseph. 1984. Phase Retrieval. *Communications in Mathematical Physics* 95: 317-343.

Rosenblatt, Joseph and P. D. Seymour. 1982 (September). The Structure of Homometric Sets. *SIAM J. Alg. Disc. Methods* 3/3: 343-350.

Satyendra, Ramon. 2004. An Informal Introduction to Some Formal Concepts from Lewin's Transformational Theory. *Journal of Music Theory* 48/1: 99-141.

Schiff, David. 1983. *The Music of Elliott Carter*. New York: Da Capo Press. Revised second edition, London: Faber; Ithaca, NY: Cornell University Press, 1998.

Schmalfeldt, Janet. 1983. *Berg's Wozzeck: Harmonic Language and Dramatic Design.* New Haven: Yale University Press.

Schillinger, Joseph. 1941. *The Schillinger System of Musical Composition,* ed. L. Dowling and A. Shaw. Reprinted ed., New York: Da Capo Press, 1976.

Schoenberg, Arnold. 1978. *The Theory of Harmony*. Translated by Roy E. Carter based on the 3rd ed. (1922). Berkeley: University of California Press.

Schoenberg, Arnold and Claudio Spies. 1965. Analysis of the Four Orchestral Songs Opus 22. *Perspectives of New Music* 3/2: 1-21.

Schuijer, Michiel. 2008. *Analyzing Atonal Music: Pitch-Class Set Theory and Its Contexts.* Rochester, NY: University of Rochester Press.

Senechal, Marjorie. 2008. A point set puzzle revisited. *European Journal of Combinatorics* 29: 1933-1944.

Soderberg, Stephen. 1995. Z-Related Sets as Dual Inversion. *Journal of Music Theory* 39/1: 77-100.

Starr, Daniel. 1978. Sets, Invariance and Partitions. *Journal of Music Theory* 22: 1-42.

Straus, Joseph. 2000. *Introduction to Post-Tonal Theory*. 2nd ed, New Jersey: Prentice Hall.

Taruskin, Richard. 1980. Russian Folk Melodies in “The Rite of Spring”. *Journal of the American Musicological Society* 33/3: 501-543.

Toussaint, Godfried. 2006. Interlocking rhythms, duration interval content, cyclotomic sets, and the hexachordal theorem. Abstract from Fourth International Workshop on Computational Music Theory, Universidad Politecnica de Madrid, Escuela Universitaria de Informatica, July 24-28.

Tymoczko, Dmitri. 2008. Set-Class Similarity, Voice Leading, and the Fourier Transform. *Journal of Music Theory* 52/2: 251-272.

Watkins, Glenn. 1995. *Soundings: Music in the Twentieth Century*. New York: Schirmer Books.

Wilcox, Howard J. 1983. Group Tables and Generalized Hexachord Theorem. *Perspectives of New Music* 21/2: 535-39.